

## ANALYTIC TWISTS OF MODULAR FORMS

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ABSTRACT. We investigate non-correlation of Fourier coefficients of Maass forms against a class of real oscillatory functions, in analogy to known results with Frobenius trace functions. We also establish an equidistribution result for twisted horocycles as a consequence of our non-correlation result.

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## 1. INTRODUCTION

In this paper we are interested in sums of Fourier coefficients of  $\mathrm{GL}_2$  Maass forms against a certain class of oscillatory functions. The type of oscillatory functions we consider can be thought as archimedean analogs of trace functions studied by Fouvry, Kowalski and Michel in [4]. Our main result gives a non-correlation statement between Fourier coefficients of Maass forms against a family of functions,  $K_t : \mathbb{R}_{>0} \rightarrow \mathbb{C}$ , depending on a large real parameter  $t$ .

**1.1. Setup.** We let throughout  $f$  be a fixed cuspidal Maass Hecke eigenform for  $\mathrm{SL}_2(\mathbb{Z})$ , and denote by  $1/4 + t_f^2$  the associated eigenvalue of the Laplacian. The form  $f$  admits a Fourier expansion

$$f(z) = \sum_{n \neq 0} \rho_f(n) |n|^{-1/2} W_{it_f}(4\pi |n| y) e(nx),$$

where  $W_\nu$  is a Whittaker function,

$$W_{it}(y) = \frac{e^{-y/2}}{\Gamma(\frac{1}{2} + it)} \int_0^\infty e^{-x} x^{it - \frac{1}{2}} \left(1 + \frac{x}{y}\right)^{it - \frac{1}{2}} dx.$$

The Fourier coefficients,  $\rho_f(n)$ , are normalized so that by Rankin-Selberg,

$$(1) \quad \sum_{n \leq X} |\rho_f(n)|^2 \asymp X.$$

We moreover know that the Fourier coefficients oscillate substantially. For example, the following estimate

$$(2) \quad \sum_{n \leq x} \rho_f(n) e(\alpha n) \ll_f x^{1/2+\epsilon}$$

holds for any  $\epsilon > 0$  uniformly for all  $\alpha \in \mathbb{R}$  (see [5] theorem 8.1). In order to understand better the oscillatory nature of the Fourier coefficients, we make the following definition.

**Definition 1.** Let  $(K(n))_{n \in \mathbb{N}}$  be a bounded sequence of complex numbers. We say that  $(K(n))$  does not correlate with  $(\rho_f(n))$  if we have

$$\sum_{n \leq x} \rho_f(n) K(n) \ll_{f,A} x(\log x)^{-A},$$

for all  $A \geq 1, x > 1$ .

Non-correlation statements are therefore a way to measure the extent to which the oscillations of a given sequence “lines up” with the oscillations of the Fourier coefficients. For example, (2) gives a non-correlation statement for the additive twist  $K(n) = e(\alpha n)$  with a power saving of  $1/2 - \epsilon$ . Another important example of non-correlation arises when  $K(n) = \mu(n)$ , the Möbius function, in which case non-correlation is an incarnation of the Prime Number Theorem (see [3] for a general result combining this and additive twists). Obtaining power saving statements against the Möbius function would be equivalent to proving a strong zero-free region towards the Riemann Hypothesis for the  $L$ -function attached to  $f$ . We give here a final example, which will be the main motivation for our work: let  $p$  be a prime number and let  $K$  be an isotypic trace function of conductor  $p$ , then [4] gives a non-correlation result for  $(K(n))$  with a power saving of  $1/8 - \epsilon$ .

We will study non-correlation against a family of functions  $(K_t)_{t \in \mathbb{R}}$ ,

$$K_t : \mathbb{R}_{>0} \rightarrow \mathbb{C},$$

where  $t$  is a parameter which we will let grow to infinity.

**Definition 2.** A family of smooth functions  $(K_t)_{t \in \mathbb{R}}, K_t : \mathbb{R}_{>0} \rightarrow \mathbb{C}$  is called a family of analytic trace functions if there exist real numbers  $a < b, b > 0$  and a family of analytic functions  $(M_t(s))_{t \in \mathbb{R}}$  in the strip  $a < \Re(s) < b$ , such that the following conditions hold.

1. The following integral converges for any  $a < \sigma < b$ ,

$$(3) \quad \frac{1}{2\pi i} \int_{(\sigma)} M_t(s) x^{-s} ds,$$

and is equal to  $K_t(x)$  for all  $x \in \mathbb{R}_{>0}, t \in \mathbb{R}$ .

2. There exist constants  $c_1, c_2$  depending on the family  $(K_t)_{t \in \mathbb{R}}$ , independent of  $t$ , such that we may write  $M_t(\sigma + i\nu) = g_t(\sigma + i\nu)e(f_t(\sigma + i\nu))$ , in such a way that for all  $x \in [t, 2t]$ , the following

$$(4) \quad g_t^{(j)}(\sigma + i\nu) \ll_j \nu^{\sigma-1/2-j} \quad \forall j \geq 0,$$

holds, as well as the following conditions on  $f_t$ .

(a) Whenever  $|\nu| \leq c_1 t$  or  $|\nu| \geq c_2 t$ , we have

$$(5) \quad \left| f_t'(\sigma + i\nu) - \frac{1}{2\pi} \log(x) \right| \gg 1.$$

(b) When  $c_1 t \leq |\nu| \leq c_2 t$ , either (5) holds, or we have

$$(6) \quad f_t''(\sigma + i\nu) \gg \nu^{-1},$$

while for all  $\epsilon > 0, j \geq 0$ ,

$$(7) \quad f_t^{(j)}(\sigma + i\nu) \ll_{j,\epsilon} \nu^{1+\epsilon-j}.$$

(c) Finally, we require that

$$(8) \quad f_t''(\sigma + i\nu) - \frac{1}{2\pi\nu} \gg \nu^{-1},$$

whenever  $c_1 t \leq |\nu| \leq c_2 t$ .

*Remark 1.* Throughout the paper, we will abuse notation and say that  $K_t$  is an analytic trace function when it arises as part of such a family.

*Remark 2.* Conditions (3) - (7) guarantee by means of stationary phase that the integral representation is concentrated around multiplicative character of conductor  $t$ . Condition (8) ensures that we avoid functions such as  $e(x)$ , as motivated in Section 5.

*Remark 3.* By the properties of the Mellin transform, we note that if  $K_t(x)$  is an analytic trace function, then for any constant  $\alpha \in \mathbb{R}_{>0}$ , we have that  $K_t(\alpha x)$  is also an analytic trace function.

*Remark 4.* We note that in interesting examples, in conjunction with condition (5), we will also have some stationary points in the region  $c_1 t \leq |\nu| \leq c_2 t$ , guaranteeing that  $\|K_t\|_\infty \asymp 1$ .

*Remark 5.* We note that in practice, we may always ensure that condition (3) holds, by studying  $K_t(x)V\left(\frac{x}{t}\right)$ , where  $V$  is a smooth compactly supported function in  $[\frac{1}{2}, 2]$ . In that case,  $M_t(s)$  is given by  $\int_0^\infty K_t(x)x^{s-1}dx$ , and the integral in (3) converges absolutely.

We give here some examples of analytic trace functions (see Section 5 for proofs).

**Example 1.** The normalized  $J$ -Bessel function of order  $t$ ,

$$F_{it}(x) := t^{1/2} \Gamma\left(\frac{1}{2} + it\right) J_{it}(x),$$

is an analytic trace function.

This should be thought of as an archimedean analog of Kloosterman sums. We now give as a second example that of higher rank Bessel functions as appearing in [9], in analogy to hyper-Kloosterman sums.

**Example 2.** For any  $n \geq 3$ , the  $n$ -th rank Bessel function of order  $t$ ,

$$J_{n,t} := \frac{t^{\frac{n-1}{2}}}{2\pi i n} \int_{(\frac{1}{4})} \Gamma\left(\frac{s - int}{n}\right) \Gamma\left(\frac{s}{n} + \frac{it}{n-1}\right)^{n-1} e\left(\frac{s}{4}\right) x^{-s} ds,$$

is an analytic trace function.

We will study sums of the shape

$$S(t) := \sum_n \rho_f(n) K_t(n) V\left(\frac{n}{t}\right),$$

where  $K_t$  is an analytic trace function and  $V$  is a smooth function supported in  $[1, 2]$  and such that  $V^{(j)}(x) \ll_j 1$ . for convenience we also normalize  $V$  so that  $\int V(y)dy = 1$ . We will show in Section 3 that any analytic trace function,  $K_t$ , satisfies  $\|K_t\|_\infty \ll 1$ , so that by Cauchy-Schwarz and (1), we have that

$$S(t) \ll t.$$

Our main result improves on that bound.

**Theorem 1.** Let  $K_t : \mathbb{R} \rightarrow \mathbb{C}$  be an analytic trace function. We have

$$S(t) \ll t^{1-1/8+\epsilon},$$

where the implicit constant depends only on  $f, \epsilon$  and on  $\|K_t\|_\infty$ .

*Remark 6.* For simplicity we have studied the case where  $n \asymp t$ . We note that for  $N \leq t$ , one may study similarly

$$Z(N) := \sum_n \rho_f(n) K_t(n) V\left(\frac{n}{N}\right).$$

If for  $x \asymp N$ , conditions (5) - (8) hold (which is the case in practice), we may show that

$$Z(N) \ll t^{1/2+\epsilon} N^{3/8},$$

which improves on the trivial bound so long as  $N \gg t^{4/5+\epsilon}$ .

Our bound has an application to the geometric question of equidistribution of horocycle flows with respect to a twisted signed measure. Let us recall that for every continuous compactly supported function  $f$  on  $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$ , we have

$$\int_0^1 f(x + iy) dx \rightarrow \mu(\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H})^{-1} \int_{\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}} f(z) d\mu(z),$$

as  $y \rightarrow 0$ , where  $\mu(z) = \frac{dx dy}{y^2}$  denotes the hyperbolic measure (see [13]). In [11] Strömbergsson gives a similar result by restricting to subsegments of hyperbolic length  $y^{-1/2-\delta}$ , i.e. that for any  $\delta > 0$  and  $f$  as above,

$$\frac{1}{\beta - \alpha} \int_\alpha^\beta f(x + iy) dx \rightarrow \mu(\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H})^{-1} \int_{\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}} f(z) d\mu(z),$$

uniformly as  $y \rightarrow 0$  so long as  $\beta - \alpha$  remains bigger than  $y^{1/2-\delta}$ . We use Theorem 1 to give the following twisted version of Strömbergsson's result, which is analogous to what is proven in [4] for horocycles twisted by Frobenius trace functions.

**Theorem 2.** *Let  $(K_t)_{t \in \mathbb{R}}$  be a family of analytic trace functions. Let  $f$  be a Maass form on  $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$ , and  $V$  be a smooth real valued function with compact support in  $[\frac{1}{2}, \frac{5}{2}]$  such that  $V^{(j)}(x) \ll 1$ , for all  $j \geq 0$ . We then have for any  $\delta > 0$ ,*

$$\frac{1}{\beta - \alpha} \int_\alpha^\beta f(x + iy) K_{1/y} \left( \frac{x}{y} \right) V(x) dx \rightarrow 0,$$

uniformly as  $y \rightarrow 0$  so long as  $\beta - \alpha$  remains bigger than  $y^{1/8-\delta}$ .

**1.2. Outline of proof of Theorem 1.** We will show in Section 3 that our definition of analytic trace function implies that we may essentially write

$$K_t(x) = \frac{1}{2\pi} \int_{\nu \asymp t} g_t(\sigma + i\nu) e(f_t(\sigma + i\nu)) x^{-\sigma - i\nu} d\nu.$$

Interchanging order of summation and integration, we may therefore write

$$S(t) = \frac{1}{2\pi} \int_{\nu \asymp t} g_t(\sigma + i\nu) e(f_t(\sigma + i\nu)) \sum_{n=1}^{\infty} \rho_f(n) n^{-\sigma - i\nu} V\left(\frac{n}{t}\right) d\nu.$$

We then adapt the circle method of Munshi, as in [8], allowing us to write the inner sum essentially as

$$\frac{1}{K} \int_K^{2K} \sum_{q \asymp Q} \sum_{\substack{a \asymp Q \\ (a,q)=1}} \frac{1}{aq} \sum_{n \asymp t} \rho_f(n) n^{iv} e\left(\frac{n\bar{a}}{q} - \frac{nx}{aq}\right) \sum_{m \asymp t} m^{-i(\nu+v)} e\left(-\frac{m\bar{a}}{q} + \frac{mx}{aq}\right) dv,$$

where  $K \leq t$  is a parameter that will ultimately be chosen optimally to be  $K = t^{1/2}$ , and  $Q = (t/K)^{1/2}$ . We may now apply Poisson summation to the  $m$ -sum, and

Voronoi summation to the  $n$ -sum to arrive at the following expression for  $S(t)$ ,

$$\sum_{n \leq K} \frac{\rho_f(n)}{\sqrt{n}} \sum_{q \asymp Q} \sum_{\substack{(m,q)=1 \\ 1 \leq |m| \leq q}} e\left(\frac{n\bar{m}}{q}\right) \int_{-K}^K \int_{\nu \asymp t} n^{-i\tau/2} g(q, m, \tau, \nu) e(f(q, m, \tau, \nu)) d\nu d\tau,$$

where  $g$  is a non-oscillatory amplitude function of size  $K$  and  $f$  is a well understood phase. In particular, we note that (8) implies that  $f''(q, m, \tau, \nu) \gg |\nu|^{-1}$ , so that we may use second derivative bounds for multivariable integrals and save in the integral. Applying the Cauchy-Schwarz inequality to get rid of the Fourier coefficients, and using the second derivative bound to save  $(Kt)^{1/2}$  in the integral, we arrive at

$$S(t) \ll Kt^{1/4} \left( \sum_{q, q' \asymp Q} \sum_{m, m' \asymp Q} \left( \frac{Q^{-2}}{K^{1/2}} + \sum_{\substack{n \asymp t \\ n \equiv q\bar{m}' - q'\bar{m} \pmod{qq'}}} \frac{1}{K^{3/2}|n|^{1/2}} \right) \right)^{1/2} \\ \ll K^{1/4} t^{3/4} + \frac{t}{K^{1/4}},$$

which upon taking  $K = t^{1/2}$  gives the desired result.

**1.3. Notations.** Throughout the paper, we will let  $f(x) \ll g(x)$ ,  $f(x) \gg g(x)$  and  $f(x) = O(g(x))$  denote the usual Vinogradov symbols. The notation  $f(x) \asymp g(x)$  will be used to mean that both  $f(x) \ll g(x)$  and  $g(x) \ll f(x)$  hold. Moreover, any subscript in these notations will be taken to mean that the implied constants are allowed to depend on those parameters. The notation  $\bar{a} \pmod{q}$  will always be used to denote the multiplicative inverse of  $a$  modulo  $q$ .

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## 2. STATIONARY PHASE INTEGRALS

Throughout the paper, we will need several stationary phase lemmas to estimate oscillatory integrals. In particular, we will regularly be faced with a special kind of oscillatory integral which we now define. Let  $W$  be any smooth real valued function, with support in  $[a, b] \subset (0, \infty)$ , and such that  $W^{(j)}(x) \ll_{a,b,j} 1$ . We then define

$$W^\dagger(r, s) := \int_0^\infty W(x) e(-rx) x^{s-1} dx,$$

where  $r \in \mathbb{R}$  and  $s \in \mathbb{C}$ . Munshi gives in [8] estimations and asymptotics for  $W^\dagger$ , however we will also need a slightly more precise version of this asymptotic. To this purpose, we quote from [1] a version of the stationary lemma.

**Lemma 1.** *Let  $0 < \delta < 1/10$ , and  $X, Y, V, V_1, Q > 0, Z := Q + X + Y + V_1 + 1$ , and assume that*

$$Y \geq Z^{3\delta}, V_1 \geq V \geq \frac{QZ^{\delta/2}}{Y^{1/2}}.$$

*Suppose that  $w$  is a smooth function on  $\mathbb{R}$  with support on an interval  $[a, b]$  of finite length  $V_1$ , satisfying*

$$w^{(j)}(t) \ll_j XV^{-j},$$

for all  $j \geq 0$ . Suppose that  $h$  is a smooth function on  $[a, b]$ , such that there exists a unique point  $t_0$  in the interval such that  $h'(t_0) = 0$ , and furthermore that

$$h''(t) \gg \frac{Y}{Q^2}, h^{(j)}(t) \ll_j \frac{Y}{Q^j}, \quad \text{for } j = 1, 2, 3, \dots, t \in [a, b].$$

Then, the integral defined by

$$I := \int_{-\infty}^{\infty} w(t) e^{ih(t)} dt$$

has an asymptotic expansion of the form

$$I = \frac{e^{ih(t_0)}}{\sqrt{h''(t_0)}} \sum_{n \leq 3\delta^{-1}A} p_n(t_0) + O_{A,\delta}(Z^{-A}),$$

and

$$(9) \quad p_n(t_0) := \frac{\sqrt{2\pi} e^{\pi i/4}}{n!} \left( \frac{i}{2h''(t_0)} \right)^n G^{(2n)}(t_0),$$

where  $A$  is arbitrary, and

$$(10) \quad G(t) := w(t) e^{iH(t)}; H(t) = h(t) - h(t_0) - \frac{1}{2} h''(t_0)(t - t_0)^2.$$

Furthermore, each  $p_n$  is a rational function in  $h', h'', \dots$ , satisfying

$$(11) \quad \frac{d^j}{dt_0^j} p_n(t_0) \ll_{j,n} X (V^{-j} + Q^{-j}) \left( (V^2 Y / Q^2)^{-n} + Y^{-n/3} \right).$$

We want to extract the first five terms in the asymptotic expansion, in order to have a small enough error term that will be easy to deal with. We therefore compute

$$p_0(t_0) = \sqrt{2\pi} e(1/8) w(t_0),$$

and

$$G'(t) = w'(t) e^{iH(t)} + i w(t) H'(t) e^{iH(t)},$$

$$G''(t) = e^{iH(t)} (w''(t) + 2i w'(t) H'(t) + i w(t) H''(t) - w(t) H'(t)^2).$$

We now see that  $H(t_0) = 0$ , while

$$H'(t) = h'(t) - h''(t_0)(t - t_0),$$

and

$$H''(t) = h''(t) - h''(t_0).$$

Hence, we see that also  $H'(t_0), H''(t_0) = 0$ . We therefore have

$$p_1(t_0) = \sqrt{2\pi} e(1/8) \frac{i}{2h''(t_0)} w''(t_0).$$

Noting that only the terms that don't contain  $H^{(i)}$  for  $i = 0, 1, 2$  survive, and that  $H^{(j)}(t) = h^{(j)}(t)$  for  $j \geq 3$ , we have

$$G^{(4)}(t_0) = w^{(4)}(t_0) + 4i w'(t_0) h^{(3)}(t_0) + i w(t_0) h^{(4)}(t_0),$$

and thus

$$p_2(t_0) = -\frac{\sqrt{2\pi} e \left(\frac{1}{8}\right)}{8h''(t_0)^2} (w^{(4)}(t_0) + 4i w'(t_0) h^{(3)}(t_0) + i w(t_0) h^{(4)}(t_0)).$$

In general,  $G^{(2n)}(t_0)$  is a linear combination of terms of the form

$$w^{(\nu_0)}(t_0) H^{(\nu_1)}(t_0) \dots H^{(\nu_l)},$$

where  $\nu_0 + \dots + \nu_l = 2n$ .

We now wish to use these in the context of the study of  $W^\dagger(r, s)$ , where we write  $s = \sigma + i\beta \in \mathbb{C}$ . We may thus use the lemma above with

$$w(x) = W(x)x^{\sigma-1},$$

and

$$h(x) = -2\pi r x + \beta \log x.$$

Then,

$$(12) \quad h'(x) = -2\pi r + \frac{\beta}{x}, \text{ and } h^{(j)}(x) = (-1)^{j-1}(j-1)! \frac{\beta}{x^j},$$

for  $j \geq 2$ . The unique stationary point is given by

$$x_0 = \frac{\beta}{2\pi r}.$$

We now let

$$\check{W}(x) := x^{1-\sigma} \sum_{n=0}^5 p_n(x),$$

and claim it is non-oscillatory in the following sense.

**Claim 1.** *Let  $\beta \gg 1$ . Then for all  $j \geq 0$ , and  $x \in [a, b]$ ,*

$$\check{W}^j(x) \ll_{\sigma, j, a, b} 1.$$

*Proof.* We compute

$$\check{W}^{(j)}(x) = \sum_{l=0}^j \binom{j}{l} (x^{1-\sigma})^{(j-l)} \sum_{n=0}^5 p_n^{(l)}(x).$$

Now, it is clear that  $(x^{1-\sigma})^{(j-l)} \ll_{j, \sigma, a, b} 1$ , and so we just need to control the derivatives of each  $p_n$ . Since  $w$  is a product of a power of  $x$  with  $W$  and  $W^{(j)}(x) \ll_j 1$ , we can easily see that  $p_0(x) \ll_{j, \sigma, a, b} 1$ . Now

$$h''(x_0) = -\frac{\beta}{x_0^2},$$

and since  $\beta \gg 1$ , by the same argument as for  $p_0$ , it is clear that  $p_1(x) \ll 1$ . We may apply the same reasoning for  $p_2$ , and more generally for any  $p_n$ , since (12) implies the higher derivatives of  $h$  don't grow compared to the powers of  $h''$  in the denominator.  $\square$

We may now give the following result for  $W^\dagger(r, s)$ .

**Lemma 2.** *Let  $r \in \mathbb{R}$  and  $s = \sigma + i\beta \in \mathbb{C}$ , such that  $x_0 = \frac{\beta}{2\pi r} \in [a/2, 2b]$ . Then,*

$$W^\dagger(r, s) = \frac{\sqrt{2\pi}e(1/8)}{\sqrt{-\beta}} \left(\frac{\beta}{2\pi r}\right)^\sigma \left(\frac{\beta}{2\pi r}\right)^{i\beta} \check{W}\left(\frac{\beta}{2\pi r}\right) + O(\min\{|\beta|^{-5/2}, |r|^{-5/2}\}).$$

*Proof.* This is a direct application of Lemma 1 with  $X = V = Q = 1, Y = \max\{|\beta|, |r|\}, V_1 = b - a$ , using the above computations as well as (11).  $\square$

We also quote from [8] the following lemma.

**Lemma 3.**

$$W^\dagger(r, s) = O_{a, b, \sigma, j} \left( \min \left\{ \left( \frac{1+|\beta|}{|r|} \right)^j, \left( \frac{1+|r|}{|\beta|} \right)^j \right\} \right).$$

3. ANALYSIS OF  $K_t$ 

In this section, we analyse further the integral representation of  $K_t$ . We make a partition of unity in the integral: let  $\mathcal{I} = \{0\} \cup_{j \geq 0} \{\pm (\frac{4}{3})^j\}$ , such that for each  $l \in \mathcal{I}$ , we take a smooth function  $W_l(x)$  supported in  $[\frac{3l}{4}, \frac{4l}{3}]$  for  $l \neq 0$  and such that

$$x^k W_l^{(k)}(x) \ll_k 1,$$

for all  $k \geq 0$ . for  $l = 0$ , take  $W_0(x)$  supported in  $[-2, 2]$  with  $W_0^{(k)}(x) \ll_l 1$ . and such that  $1 = \sum_{l \in \mathcal{I}} W_l(x)$ . We then let for any  $i \in \mathcal{I}$ ,

$$I_{l,t}(x) := \frac{1}{2\pi} \int_{\mathbb{R}} g_t(\sigma + i\nu) e(f_t(\sigma + i\nu)) x^{-\sigma - i\nu} W_l(\nu) d\nu.$$

We prove the following result.

**Lemma 4.** *Let  $K_t$  be an analytic trace function. We have, for  $x \in [t, 2t]$ , and any  $\epsilon > 0$ ,*

$$K_t(x) = \sum_{\text{Supp}(W_l) \subset [\pm t^{1-\epsilon}, \pm t^{1+\epsilon}] \cup [-t^\epsilon, t^\epsilon]} I_{l,t}(x) + O(t^{-1000}).$$

Moreover, we also have

$$\max_{x \in [t, 2t]} |K_t(x)| \ll 1.$$

*Proof.* Condition (3) implies that we may write

$$(13) \quad K_t(x) = \frac{1}{2\pi i} \int_{(\sigma)} M_t(s) x^{-s} ds = \frac{1}{2\pi} \int_{\mathbb{R}} g_t(\sigma + i\nu) e(f_t(\sigma + i\nu)) x^{-\sigma - i\nu} d\nu,$$

for any  $\sigma \in [a, b]$ . We now wish to run a stationary phase argument to localise the integral around the points without too much oscillation. If  $l \ll t^\epsilon$  for some small  $0 < \epsilon < \sigma/(1/2 + \sigma)$ , then

$$I_{l,t}(x) \ll t^{\epsilon + (\sigma - 1/2) - \sigma} = o(1),$$

as long as we take  $\sigma > 0$ . We now fix such an  $\epsilon$  and look at  $l$  such that  $\text{Supp}(W_l) \subset [\pm t^\epsilon, \pm \infty)$ , and look at

$$x^\sigma I_{l,t}(x) = \int_{\mathbb{R}} g_t(\sigma + i\nu) W_l(\nu) e\left(f_t(\sigma + i\nu) - \frac{\nu}{2\pi} \log(x)\right) d\nu,$$

for  $x \in [t, 2t]$ . We now compute a few derivatives, in order to apply stationary phase arguments. We have by (4)

$$(g_t(\sigma + i\nu) W_l(\nu))^{(j)}(\nu) \ll_j i^{\sigma - 1/2 - j}, \quad \forall j \geq 0,$$

while by (5)

$$f'_t(\sigma + i\nu) - \frac{\log(x)}{2\pi} \gg 1,$$

if  $\nu \neq t$  and by (7)

$$f_t^{(j)}(\sigma + i\nu) \ll l^{1 + \epsilon/2 - j}.$$

Therefore, in the case that  $\nu \neq t$ , we may use Lemma 1 (with  $X = l^{\sigma - 1/2}$ ,  $U = l$ ,  $\beta - \alpha = 3l/2$ ,  $R = 1$ ,  $Y = l^{1 + \epsilon/2}$  and  $Q = l$ ), to deduce that

$$I_{l,t}(x) \ll_A l^{-A},$$

for any  $A > 0$ .

In the case that  $\nu \asymp t$ , we use the second derivative bound for oscillatory integrals along with (6) to deduce that

$$I_{l,t}(x) \ll 1.$$

□



To conclude this section we note that the case where  $\text{Supp}(W_l) \subset [-t^\epsilon, t^\epsilon]$  can be handled as follows. Since  $V$  is a smooth compactly supported function, it admits a Mellin transform,

$$\tilde{V}(s) = \int_0^\infty V(x)x^{s-1}dx,$$

that decays very rapidly in vertical strips. One can thus write for any  $\alpha \in \mathbb{R}$ ,

$$V(x) = \int_{(\alpha)} \tilde{V}(s)x^{-s}ds.$$

Using this, we write for any  $\sigma \geq 0$ ,

$$\begin{aligned} \sum_{n=1}^\infty \rho_f(n) I_{l,t}(n) V\left(\frac{n}{t}\right) &= \int_{\mathbb{R}} M_t(\sigma + i\nu) W_l(\nu) \sum_{n=1}^\infty \rho_f(n) n^{-\sigma - i\nu} V\left(\frac{n}{t}\right) d\nu \\ &= \int_{\mathbb{R}} \int_{(\alpha)} M_t(\sigma + i\nu) W_l(\nu) \tilde{V}(s) t^s L(f, \sigma + i\nu + s) ds d\nu \\ &\ll t^{1/2+\epsilon}, \end{aligned}$$

by the rapid decay of  $\tilde{V}$ .

We will therefore only focus on the cases where the support of  $W_l$  is close to  $t$ . This may be interpreted as the fact that the spectral decomposition of any analytic trace function,  $K_t$ , concentrates around multiplicative characters of conductor  $t$ .

#### 4. PROOF OF THEOREM 1

Following Munshi [8] we adapt Kloosterman's version of the circle method along with a conductor dropping mechanism. We quote here the following proposition in [6].

**Proposition 1.** *Let*

$$\delta(n) = \begin{cases} 1 & \text{if } n = 0; \\ 0 & \text{otherwise.} \end{cases}$$

*Then, for any real number  $Q \geq 1$ , we have*

$$\delta(n) = 2\Re \int_0^1 \sum_{1 \leq q \leq Q < a \leq q+Q}^* \frac{1}{aq} e\left(\frac{n\bar{a}}{q} - \frac{nx}{aq}\right) dx.$$

In particular, we will use this proposition with  $Q := (t/K)^{1/2}$ , where  $t^{\epsilon'} < K < t^{1-\epsilon'}$  (for some  $\epsilon' > 0$ ) is a parameter to be chosen optimally later. We let

$$S_l(t) := \sum_{n=1}^\infty \rho_f(n) I_{l,t}(n) V\left(\frac{n}{t}\right),$$

and note that in order to bound non-trivially  $S(t)$ , it is sufficient to do so for  $S_l(t)$ , for  $i$  such that  $\text{Supp} W_l \subset [\pm t^{1-\epsilon}, \pm t^{1+\epsilon}]$ , as follows from the previous section. We may thus write

$$\begin{aligned} S_l(t) &= \sum_{n=1}^\infty \rho_f(n) I_{l,t}(n) V\left(\frac{n}{t}\right) \\ &= \frac{1}{K} \int_{\mathbb{R}} V\left(\frac{v}{K}\right) \sum_{n,m=1}^\infty \rho_f(n) I_{l,t}(m) \left(\frac{n}{m}\right)^{iv} V\left(\frac{n}{t}\right) U\left(\frac{m}{t}\right) dv \\ &= S_l^+(t) + S_l^-(t), \end{aligned}$$

where  $U$  is a smooth functions supported in  $[1/2, 5/2]$ , with  $U(x) = 1$  for  $x \in \text{Supp}(V)$  and  $U^{(j)} \ll_j 1$ , and

$$S_l^\pm(t) = \frac{1}{K} \int_0^1 \int_{\mathbb{R}} V\left(\frac{v}{K}\right) \sum_{1 \leq q \leq Q < a \leq Q+q}^* \frac{1}{aq} \\ \times \sum_{n,m=1}^{\infty} \rho_f(n) n^{iv} I_{l,t}(m) m^{-iv} e\left(\pm \frac{(n-m)\bar{a}}{q} \mp \frac{(n-m)x}{aq}\right) V\left(\frac{n}{t}\right) U\left(\frac{m}{t}\right) dv dx.$$

We will now describe the analysis for  $S_l^+(t)$  (the analysis for  $S_l^-(t)$  being completely analogous).

**4.1. Summation formulae.** We start with the  $m$ -sum, which we split into congruence classes mod  $q$ , and after applying Poisson summation, we obtain

$$\sum_{m=1}^{\infty} I_{l,t}(m) m^{-iv} U\left(\frac{m}{t}\right) e\left(-\frac{m\bar{a}}{q}\right) e\left(\frac{mx}{aq}\right) \\ = \sum_{\substack{m \in \mathbb{Z} \\ m \equiv \bar{a} \pmod{q}}} \frac{t^{1-\sigma-iv}}{2\pi} \int_{\mathbb{R}} t^{-i\nu} M_t(\sigma + i\nu) W_l(\nu) U^\dagger\left(\frac{t(ma-x)}{aq}, 1-\sigma-i(\nu+v)\right) d\nu.$$

We now note that since  $|\nu| \in [t^{1-\epsilon}, t^{1+\epsilon}]$ , we may as in [8] use Lemma 3 to deduce that only the contribution from  $1 \leq |m| \ll qt^\epsilon$  is non-negligible. We take a dyadic subdivision to obtain the following.

**Lemma 5.**

$$S_l^+(t) = \frac{t^{1-\sigma}}{K} \sum_{1 \leq C \leq (t/K)^{1/2}} S_l(t, C) + O(t^{-1000}),$$

where  $C$  runs over dyadic integers and

$$S_l(t, C) = \frac{1}{2\pi} \int_0^1 \int_{\mathbb{R}} M_t(\sigma + i\nu) W_l(\nu) t^{-i(v+\nu)} V\left(\frac{v}{K}\right) \sum_{C < q \leq 2C} \sum_{\substack{(m,q)=1 \\ 1 \leq |m| \ll qt^\epsilon}} \frac{1}{aq} \\ \times U^\dagger\left(\frac{t(ma-x)}{aq}, 1-\sigma-i(v+\nu)\right) \sum_{n=1}^{\infty} \rho_f(n) n^{iv} e\left(\frac{nm}{q}\right) e\left(-\frac{nx}{aq}\right) V\left(\frac{n}{t}\right) dv dx d\nu$$

and  $a = a_Q(m, q)$  is the unique multiplicative inverse of  $m \pmod{q}$  in  $(Q, q+Q]$ .

We wish to use the Voronoi summation on the  $n$ -sum. We quote from [7] the following formula.

**Lemma 6.** *Let  $g$  be a Hecke-Maass form over  $\text{SL}_2(\mathbb{Z})$  and spectral parameter  $t_g$ . Let  $F$  be a smooth function decaying at infinity, which vanishes in a neighborhood of the origin. Then, for  $(a, c) = 1$ , we have*

$$\sum_{n \geq 1} \rho_g(n) e\left(\frac{an}{c}\right) F(n) = \frac{1}{c} \sum_{\pm} \sum_{n \geq 1} \rho_f(\mp n) e\left(\pm \frac{n\bar{a}}{c}\right) V^\pm\left(\frac{n}{c^2}\right),$$

where

$$V^-(y) = \int_0^\infty F(x) J_g(4\pi\sqrt{xy}) dx \\ V^+(y) = \int_0^\infty F(x) K_g(4\pi\sqrt{xy}) dx,$$

and

$$J_g(x) = -\frac{\pi}{\sin(\pi i t_g)} (J_{2it_g}(x) - J_{-2it_g}(x)),$$

and

$$K_g(x) = 4 \cos(\pi i t_g) K_{2it_g}(x).$$

We now use [2, p. 326, 331] that

$$\begin{aligned} K_{2ir}(x) &= \frac{1}{4} \frac{1}{2\pi i} \int_{(\sigma')} \left(\frac{x}{2}\right)^{-s} \Gamma\left(\frac{s}{2} + ir\right) \Gamma\left(\frac{s}{2} - ir\right) ds, & |\Re(2ir)| < \sigma' \\ J_{2ir}(x) &= \frac{1}{2} \frac{1}{2\pi i} \int_{(\sigma')} \left(\frac{x}{2}\right)^{-s} \frac{\Gamma(s/2 + ir)}{\Gamma(1 - s/2 + ir)} ds, & -\Re(2ir) < \sigma' < 1, \end{aligned}$$

and define

$$\begin{aligned} \gamma_-(s) &= \frac{-\pi}{4\pi i \sin(\pi i t_g)} \left\{ \frac{\Gamma(s/2 + it_g)}{\Gamma(1 - s/2 + it_g)} - \frac{\Gamma(s/2 - it_g)}{\Gamma(1 - s/2 - it_g)} \right\} \\ \gamma_+(s) &= \frac{4 \cos(\pi i t_g)}{8\pi i} \Gamma\left(\frac{s}{2} + it_g\right) \Gamma\left(\frac{s}{2} - it_g\right) \end{aligned}$$

to deduce that for any  $0 < \sigma' < 1$ ,

$$V^-(y) = \int_0^\infty F(x) \int_{(\sigma')} (2\pi\sqrt{xy})^{-s} \gamma_-(s) ds dx,$$

and

$$V^+(y) = \int_0^\infty F(x) \int_{(\sigma')} (2\pi\sqrt{xy})^{-s} \gamma_+(s) ds dx.$$

We are now ready to plug all of this into effect.

$$\sum_{n \geq 1} \rho_f(n) e\left(\frac{nm}{q}\right) n^{iv} e\left(\frac{-nx}{aq}\right) V\left(\frac{n}{t}\right) = \frac{t^{1+iv}}{q} \sum_{\pm} \sum_{n \geq 1} \rho_f(\mp n) e\left(\pm \frac{n\bar{m}}{q}\right) I(n, q, v, x),$$

where

$$\begin{aligned} I(n, q, v, x) &= \int_{(\sigma')} \left(\frac{2\pi\sqrt{nt}}{q}\right)^{-s} \gamma_{\pm}(s) \int_0^\infty y^{iv} e\left(\frac{-tyx}{aq}\right) V(y) y^{-s/2} dy ds \\ &= \int_{(\sigma')} \left(\frac{2\pi\sqrt{nt}}{q}\right)^{-s} \gamma_{\pm}(s) V^\dagger\left(\frac{tx}{aq}, 1 + iv - s/2\right) ds. \end{aligned}$$

By Stirling's formula:

$$\begin{aligned} \Gamma(\sigma' + it) &= \sqrt{2\pi} \exp\left(\frac{-\pi|t|}{2}\right) |t|^{\sigma'-1/2} \left|\frac{t}{e}\right|^{it} \exp(\text{sign}(t)i\pi(\sigma' - 1/2)/2)(1 + O(|t|^{-1})), \end{aligned}$$

for  $|t| \geq 1$  and bounded  $\sigma'$ , we deduce that

$$\gamma_{\pm}(\sigma' + i\tau) \ll 1 + |\tau|^{\sigma'-1}.$$

Now, by Lemma 3,

$$V^\dagger\left(\frac{tx}{aq}, 1 + iv - s/2\right) \ll \min\left\{1, \left(\frac{(Kt)^{1/2}}{|v - \tau/2|q}\right)^j\right\}.$$

Thus, shifting the contour to  $\sigma' = M$  a large positive integer and taking  $j = M + 1$  for instance, we see that if  $n \gg Kt^\epsilon$ , then the integral is negligible (by splitting the integral into a box around  $|v - \frac{\tau}{2}|q \leq (Kt)^{1/2}$  and its complement). In the remaining range, we study this more closely. We shift our contour to  $\sigma = 1$  (the  $\gamma_+$  contribution is trivial, so we only consider  $\gamma_-$ ), and note that

$$\gamma_-(i\tau + 1) = \left(\frac{|\tau|}{2e}\right)^{i\tau} \Phi_-(\tau),$$

where  $\Phi'_-(\tau) \ll |\tau|^{-1}$ . We thus have

$$I(n, q, v, x) = \frac{qi}{2\pi\sqrt{nt}} \sum_{J \in \mathcal{J}} \int_{\mathbb{R}} \left( \frac{2\pi\sqrt{nt}}{q} \right)^{-i\tau} \gamma_{\pm}(i\tau + 1) V^{\dagger} \left( \frac{tx}{aq}, \frac{1}{2} + i(v - \tau/2) \right) W_J(\tau) d\tau \\ + O(t^{-1000}),$$

where  $\mathcal{J}$  is a collection of  $O(\log t)$  integers such that  $J \in \mathcal{J}$  if and only if  $\text{Supp} W_J \subset [-(tK)^{1/2}t^{\epsilon}/C, (tK)^{1/2}t^{\epsilon}/C]$ . We have proven the following:

**Lemma 7.**

$$S_l(t, C) = \frac{iKt^{1/2}}{4\pi^2} \sum_{\pm} \sum_{J \in \mathcal{J}} \sum_{n \leq Kt^{\epsilon}} \frac{\rho_f(\mp n)}{\sqrt{n}} \sum_{C < q \leq 2C} \sum_{\substack{(m, q)=1 \\ 1 \leq |m| \leq qt^{\epsilon}}} \frac{e\left(\pm \frac{n\bar{m}}{q}\right)}{aq} I_{\pm}^*(q, m, n) \\ + O(t^{-10000}),$$

where

$$I_{\pm}^*(q, m, n) = \int_{\mathbb{R}^2} M_t(\sigma + i\nu) W_t(\nu) t^{-i\nu} \left( \frac{2\pi\sqrt{nt}}{q} \right)^{-i\tau} \gamma_{\pm}(i\tau + 1) I^{**}(q, m, \tau, \nu) W_J(\tau) d\tau d\nu,$$

and

$$I^{**}(q, m, \tau, \nu) = \int_0^1 \int_{\mathbb{R}} V(v) V^{\dagger} \left( \frac{tx}{aq}, i \left( kv - \frac{\tau}{2} \right) + \frac{1}{2} \right) \\ \times U^{\dagger} \left( \frac{t(ma - x)}{aq}, 1 - \sigma - i(Kv + \nu) \right) dv dx.$$

In the next two subsections we evaluate  $I^{**}(q, m, \tau, \nu)$ .

**4.2. Analysis of the integrals.** We apply lemma 2 to

$$U^{\dagger} \left( \frac{t(ma - x)}{aq}, 1 - \sigma - i(Kv + \nu) \right) \\ = e \left( \frac{1}{8} \right) \left( \frac{Kv + \nu}{2\pi} \right)^{1/2 - \sigma} \left( \frac{aq}{t(x - ma)} \right)^{1 - \sigma} \left( \frac{(Kv + \nu)aq}{2\pi et(x - ma)} \right)^{-i(Kv + \nu)} \\ \times \check{U} \left( \frac{(Kv + \nu)aq}{2\pi t(x - ma)} \right) + O(t^{-5/2}).$$

Hence,

$$I^{**}(q, m, \tau, \nu) = c_1 \int_0^1 \int_{\mathbb{R}} V(v) V^{\dagger} \left( \frac{tx}{aq}, i \left( Kv - \frac{\tau}{2} \right) + \frac{1}{2} \right) (Kv + \nu)^{1/2 - \sigma} \left( \frac{aq}{t(x - ma)} \right)^{1 - \sigma} \\ \times \left( \frac{(Kv + \nu)aq}{2\pi et(x - ma)} \right)^{-i(Kv + \nu)} \check{U} \left( \frac{(Kv + \nu)aq}{2\pi t(x - ma)} \right) dv dx + O(t^{-5/2}),$$

for some constant  $c_1$ . We now use lemma 5 of [8] to

$$V^{\dagger} \left( \frac{tx}{aq}, i(Kv - \tau/2) + \frac{1}{2} \right) = \frac{(aq)^{1/2} e(-\frac{1}{8})}{(tx)^{1/2}} \left( \frac{(Kv - \frac{\tau}{2})aq}{2e\pi tx} \right)^{i(Kv - \frac{\tau}{2})} V \left( \frac{(Kv - \tau/2)aq}{2\pi tx} \right) \\ + O \left( \min \left\{ |Kv - \tau/2|^{-3/2}, \left( \frac{tx}{aq} \right)^{-3/2} \right\} \right).$$

Hence,

$$I^{**}(q, m, \tau, \nu) = c_2 \int_0^1 \int_{\mathbb{R}} V(v) V \left( \frac{(Kv - \frac{\tau}{2})aq}{2\pi tx} \right) \left( \frac{(Kv - \frac{\tau}{2})aq}{2\pi tx} \right)^{i(Kv - \frac{\tau}{2})} \check{U} \left( \frac{(kv + \nu)aq}{2\pi t(x - ma)} \right) \\ \left( \frac{aq}{t} \right)^{\frac{3}{2} - \sigma} \frac{(\nu + Kv)^{\frac{1}{2} - \sigma}}{(x - ma)^{1 - \sigma}} \left( \frac{(Kv + \nu)aq}{2\pi et(x - ma)} \right)^{-i(Kv + \nu)} dv \frac{dx}{x^{1/2}} + E + O(t^{-\frac{5}{2}}),$$

for some constant  $c_2$  and where  $E$  comes from the error term of  $V^\dagger$  which we will now describe. We first note that since  $V^\dagger \left( \frac{tx}{aq}, i(Kv - \tau/2) + \frac{1}{2} \right)$  does not depend on  $\nu$ , neither does the error term, and therefore we may perform the  $\nu$ -integral without losing control of the phase, before plugging absolute values. We thus estimate

$$\begin{aligned} & \int_{\mathbb{R}} M_t(\sigma + i\nu) W_l(\nu) t^{-i\nu} (Kv + \nu)^{1/2-\sigma} \left( \frac{(Kv + \nu)aq}{2\pi et(x - ma)} \right)^{-i(Kv + \nu)} \check{U} \left( \frac{(Kv + \nu)aq}{2\pi t(x - ma)} \right) d\nu \\ &= \int_{\mathbb{R}} g(\nu) e(f(\nu)) d\nu, \end{aligned}$$

where, temporarily, we define

$$g(\nu) = g_t(\sigma + i\nu) W_l(\nu) (Kv + \nu)^{1/2-\sigma} \check{U} \left( \frac{(Kv + \nu)aq}{2\pi t(x - ma)} \right),$$

and

$$2\pi f(\nu) = 2\pi f_t(\sigma + i\nu) - \nu \log t - (Kv + \nu) \log \left| \frac{(Kv + \nu)aq}{2\pi et(x - ma)} \right|.$$

We have

$$2\pi f''(\nu) = 2\pi f_t''(\sigma + i\nu) - \frac{1}{Kv + \nu} \gg \nu^{-1},$$

by (8). Noting that  $g(\nu) \ll 1$ , and  $\int |g'(\nu)| \ll t^\epsilon$ , we may use the second derivative bound for oscillatory integrals (see [10], Lemma 5) to deduce that

$$(14) \quad \int_{\mathbb{R}} g(\nu) e(f(\nu)) d\nu \ll t^{1/2+\epsilon}.$$

Our error term,  $E$ , therefore satisfies

$$\begin{aligned} & \int_{\mathbb{R}} M_t(\sigma + i\nu) W_l(\nu) t^{-i\nu} E d\nu \\ & \ll t^{\sigma-1/2+\epsilon} \int_0^1 \int_1^2 \min \left\{ \left| Kv - \frac{\tau}{2} \right|^{-3/2}, \left( \frac{tx}{aq} \right)^{-3/2} \right\} dv dx. \end{aligned}$$

This integral is the same than the one appearing in [8], where it is proved that

$$\int_0^1 \int_1^2 \min \left\{ \left| Kv - \frac{\tau}{2} \right|^{-3/2}, \left( \frac{tx}{aq} \right)^{-3/2} \right\} dv dx \ll \frac{1}{K^{3/2}} \min \left\{ 1, \frac{10K}{|\tau|} \right\} t^\epsilon.$$

Moreover, we note that

$$\int_{\mathbb{R}} g_t(\sigma + i\nu) W_l(\nu) t^{-i\nu} \ll t^{-2+\sigma},$$

and thus (keeping in mind that  $t^\epsilon < K < t^{1-\epsilon}$ ),

$$\int_{\mathbb{R}} M_t(\sigma + i\nu) W_l(\nu) t^{-i\nu} (E + O(t^{-5/2})) d\nu \ll \frac{t^{\sigma+\epsilon}}{t^{1/2} K^{3/2}} \min \left\{ 1, \frac{10K}{|\tau|} \right\}.$$

We now treat the main term. Let  $\delta' > 0$  to be determined later and examine the contribution from  $x < 1/K^{1-\delta'}$ . Using (14) and that  $u^\alpha \check{U}(u), v^\alpha V(v) \ll 1$ , for all  $\alpha \in \mathbb{R}$ , (and thus  $t(x - ma)(aq)^{-1} \gg t^{1-\epsilon}$ ), we estimate

$$\begin{aligned} & \left( \frac{aq}{t} \right)^{1/2} \int_0^{K^{\delta'-1}} \int_{\mathbb{R}} V(v) V \left( \frac{(Kv - \frac{\tau}{2})aq}{2\pi tx} \right) \left( \frac{aq}{t(x - ma)} \right)^{1-\sigma} \left| \int_{\mathbb{R}} g(\nu) e(f(\nu)) d\nu \right| dv \frac{dx}{x^{1/2}} \\ & \ll t^\epsilon \int_0^{K^{\delta'-1}} \int_{Kv - \frac{\tau}{2} \asymp \frac{tx}{aq}} V(v) \frac{t^\sigma}{t^{1/2} (Kv - \frac{\tau}{2})^{1/2}} dv dx \ll \frac{t^{1/2+\sigma+\epsilon}}{K^{3-\epsilon} aq}, \end{aligned}$$

upon taking  $\delta' = 2\epsilon/3$ . We now look at the contribution from  $x \in [K^{\delta'-1}, 1]$ . We now reset temporarily

$$g(v) = (\nu + Kv)^{1/2-\sigma} \left( \frac{aq}{t(x-ma)} \right)^{1-\sigma} V(v) V \left( \frac{(Kv - \frac{\tau}{2})aq}{2\pi tx} \right) \check{U} \left( \frac{(Kv + \nu)aq}{2\pi t(x-ma)} \right),$$

and

$$f(v) = \frac{Kv - \frac{\tau}{2}}{2\pi} \log \left( \frac{(Kv - \frac{\tau}{2})aq}{2e\pi tx} \right) - \frac{Kv + \nu}{2\pi} \log \left( \frac{(Kv + \nu)aq}{2\pi et(x-ma)} \right).$$

Then,

$$f'(v) = -\frac{K}{2\pi} \log \left( \frac{(\nu + Kv)x}{(Kv - \frac{\tau}{2})(x-ma)} \right), f^{(j)}(v) = -\frac{(j-2)!(-K)^j}{2\pi(\nu + Kv)^{j-1}} + \frac{(j-2)!(-K)^j}{2\pi(Kv - \frac{\tau}{2})^{j-1}},$$

for  $j \geq 2$ , and the stationary point is given by

$$v_0 = -\frac{(2\nu + \tau)x - \tau ma}{2Kma}.$$

Now, since  $\nu \gg t^{1-\epsilon}$ , we have that in the support of the integral,

$$f^{(j)} \asymp \frac{tx}{aq} \left( \frac{Kaq}{tx} \right)^j,$$

for  $j \geq 2$ , and

$$g^{(j)}(v) \ll t^{-1/2+\epsilon} \left( 1 + \frac{Kaq}{tx} \right)^j,$$

for  $j \geq 0$ . Moreover, we can write

$$f'(v) = \frac{K}{2\pi} \log \left( 1 + \frac{K(v_0 - v)}{\nu + Kv} \right) - \frac{K}{2\pi} \log \left( 1 + \frac{K(v_0 - v)}{Kv - \tau/2} \right),$$

and note that in the support of the integral we have  $0 \leq Kv - \tau/2 \ll tx/aq \ll K^{1/2}t^{1/2}$ . It follows that if  $v_0 \notin [.5, 3]$ , then in the support of the integral we have

$$|f'(v)| \gg K \min \left\{ 1, \frac{Kaq}{tx} \right\}.$$

We now use Lemma 8.1 of [1] with

$$X = t^{-1/2+\epsilon}, U(=V) = \min \left\{ 1, \frac{tx}{Kaq} \right\}, \quad R = K \min \left\{ 1, \frac{Kaq}{tx} \right\}, \\ Y = \frac{tx}{aq}, \quad Q = \frac{tx}{Kaq},$$

so that, choosing  $K > t^{1/3+\epsilon}$ ,

$$\int_{\mathbb{R}} g(v) e(f(v)) dv \ll t^{-1/2+\epsilon} \left[ \left( \left( \frac{tx}{aq} \right)^{1/2} \min \left\{ 1, \frac{Kaq}{tx} \right\} \right)^{-A} + K^{-A} \right] \\ \ll t^{-1/2+\epsilon} \left[ (t^{3\epsilon/4})^{-A} + (K^{\delta'})^{-A} + K^{-A} \right] \ll t^{-B},$$

for any  $B > 0$ . In the case where  $v_0 \in [.5, 3]$ , we will use Lemma 1, with  $\delta = 1/100$ ,  $A = 10000\delta'^{-1}$  and the same  $X, Y, V$  and  $Q$  as above. We have

$$\int_{\mathbb{R}} g(v) e(f(v)) dv = \frac{e(f(v_0))}{\sqrt{2\pi f''(v_0)}} \sum_{n=0}^{300A} p_n(v_0) + O_{\delta'} \left( \left( \frac{tx}{aq} \right)^{-A} \right)$$

where  $p_n$  is given by (9). Now, since  $x \in [K^{\delta'-1}, 1]$ , we have  $tx/aq \gg K^{\delta'}$ , and therefore the error term is negligible.

4.3. **Contribution from  $n \geq 1$  terms.** We find that

$$f(v_0) = -\frac{\nu + \tau/2}{2\pi} \log \left( -\frac{(\nu + \tau/2)q}{2e\pi tm} \right),$$

and

$$(15) \quad f''(v_0) = \frac{K^2(ma)^2}{2\pi(\nu + \tau/2)(x - ma)x},$$

and

$$(16) \quad f^{(j)}(v_0) = \frac{(j-2)!(-K)^j(ma)^{j-1}((x - ma)^{j-1} + (-x)^{j-1})}{2\pi(\nu + \tau/2)^{j-1}(ma - x)^{j-1}x^{j-1}}.$$

We also find

$$(17) \quad g(v_0) = \left( \frac{tm}{(\nu + \frac{\tau}{2})q} \right)^\sigma \left( \frac{aq}{t} \right) \left( \frac{-(\nu + \frac{\tau}{2})}{(x - ma)ma} \right)^{1/2} \\ \times V \left( \frac{\tau}{2K} - \frac{(\nu + \frac{\tau}{2})x}{Kma} \right) \tilde{U} \left( \frac{-(\nu + \frac{\tau}{2})q}{2\pi tm} \right) V \left( -\frac{(\nu + \frac{\tau}{2})q}{m2\pi t} \right).$$

We wish to keep the term  $n = 0$  and show that the terms with  $n \geq 1$  can be absorbed into an error term. We thus look to bound

$$\int_{\mathbb{R}} M_t(\sigma + i\nu) W_l(\nu) t^{-i\nu} \frac{e(f(v_0))}{\sqrt{f''(v_0)}} p_n(v_0) d\nu = \int_{\mathbb{R}} \tilde{g}_n(\nu) e(\tilde{f}(\nu)) d\nu,$$

where

$$\tilde{g}_n(\nu) := \frac{\sqrt{2\pi}(x - ma)^{1/2}x^{1/2}}{Kma} g_t(\sigma + i\nu) W_l(\nu) (\nu + \tau/2)^{1/2} p_n \left( -\frac{(2\nu + \tau)x - \tau ma}{2Kma} \right),$$

and

$$\tilde{f}(\nu) := f_t(\sigma + i\nu) - \frac{\nu}{2\pi} \log t - \frac{\nu + \frac{\tau}{2}}{2\pi} \log \left( -\frac{(\nu + \frac{\tau}{2})q}{2e\pi tm} \right).$$

We compute

$$\tilde{f}'(\nu) = f'_t(\sigma + i\nu) - \frac{\log t}{2\pi} - \frac{1}{2\pi} \log \left( -\frac{(\nu + \frac{\tau}{2})q}{2e\pi tm} \right) - \frac{1}{2\pi},$$

and

$$\tilde{f}''(\nu) = f''_t(\sigma + i\nu) - \frac{1}{2\pi(\nu + \frac{\tau}{2})}.$$

In order to estimate the size of  $\tilde{g}_n$ , we estimate first

$$p_1(v_0) \ll \frac{g''(v_0)}{f''(v_0)} \ll \frac{XQ^2}{V^2Y}, \quad p_2(v_0) \ll \frac{XQ^4}{V^4Y^2} + \frac{XQ}{VY} + \frac{X}{Y},$$

while, by (11), for  $n \geq 3$  we have

$$p_n(v_0) \ll X \left( \left( \frac{V^2Y}{Q^2} \right)^{-n} + Y^{-n/3} \right).$$

We now distinguish two cases. If  $x \leq \frac{Kaq}{t}$ , then  $V = Q = \frac{tx}{Kaq}$ , and thus

$$p_n(v_0) \ll \frac{X}{Y},$$

for all  $n \geq 1$ , since  $Y = \frac{tx}{aq} \gg K^{\delta'}$ . We then show by (11) that

$$\tilde{g}_n'(\nu) \ll \frac{(x - ma)^{1/2}x^{1/2}(\nu + \frac{\tau}{2})^{1/2}X}{Kma\nu^{3/2-\sigma}Y},$$

so that by the second derivative bound for oscillatory integrals (using that  $q \asymp m$ , by the support of  $\check{U}$ ),

$$\int_{\mathbb{R}} \tilde{g}_n(\nu) e(f(\nu)) d\nu \ll \frac{t^{\sigma+\epsilon} (aq)^{1/2}}{K t x^{1/2}}.$$

Therefore the total contribution from this part is dominated by

$$\left(\frac{aq}{t}\right)^{1/2} \int_{K^{\delta'-1}}^1 \frac{t^{\sigma+\epsilon} (aq)^{1/2}}{K t x} dx \ll \frac{t^{\sigma+\epsilon}}{K^2 t^{1/2}}.$$

For  $x > \frac{K a q}{t}$ , we have  $V = 1$ , and so

$$p_n(v_0) \ll \frac{t^{1/2+\epsilon} x}{K^2 a q}.$$

In this region, we first pass the  $x$  integral inside the  $\nu$ -integral, and since the phase does not depend on  $x$ , the same analysis holds, replacing  $\tilde{g}_n(\nu)$  by

$$\hat{g}_n(\nu) := \left(\frac{aq}{t}\right)^{1/2} \int_{\max\{K^{-1+\delta'}, K a q/t\}}^1 \frac{1}{\sqrt{x}} \tilde{g}_n(\nu) dx.$$

We have, using that  $m \asymp q$ ,

$$\hat{g}_n(\nu) \ll \left(\frac{aq}{t}\right)^{1/2} \tilde{g}_n(\nu) \ll \frac{t^{\sigma+\epsilon}}{K^3 a q}.$$

In order to control  $\hat{g}_n'(\nu)$ , we will first execute the  $x$ -integral, using integration by parts. Looking at the definition of  $p_n$ , we note that it is a rational function in  $f''(v_0), f'''(v_0), \dots, g(v_0), g'(v_0), \dots$  and will describe what the terms of  $p_n$  depending on  $x$  look like. We first recall that by (9) and (10),

$$p_n(v_0) = \frac{\sqrt{2\pi} e^{\pi i/4}}{n!} \left(\frac{i}{2\tilde{f}''(v_0)}\right)^n G^{(2n)}(v_0),$$

where  $G^{(2n)}(v_0)$  is a linear combination of elements of the form

$$\hat{g}_n^{(l_0)}(v_0) \tilde{f}^{(l_1)}(v_0) \dots \tilde{f}^{(l_j)}(v_0),$$

where  $l_0 + \dots + l_j = 2n$ . Using (15), (16) and (17), we therefore have that those terms of  $p_n$  depending on  $x$  are of the shape

$$x^i (x - ma)^{j+1/2} V^{(l)} \left( \frac{\tau}{2K} - \frac{(\nu + \frac{\tau}{2})x}{Kma} \right),$$

for some  $i, j \geq 1$  and  $l \geq 0$ . We thus compute

$$\begin{aligned} & \frac{d}{d\nu} \int_{\max\{K^{-1+\delta'}, K a q/t\}}^1 x^{i-1/2} (x - ma)^{j+1/2} V^{(l)} \left( \frac{\tau}{2K} - \frac{(\nu + \tau/2)x}{Kma} \right) dx \\ &= \frac{d}{d\nu} \left( \left[ x^{i-1/2} (x - ma)^{j+1/2} \frac{-Kma}{t\nu + \tau/2} V^{(l-1)} \left( \frac{\tau}{2K} - \frac{(\nu + \tau/2)x}{Kma} \right) \right]_{\max\{K^{-1+\delta'}, K a q/t\}}^1 \right. \\ & \quad \left. + \int_{\max\{K^{-1+\delta'}, K a q/t\}}^1 (x^{i-1/2} (x - ma)^{j+1/2})' \frac{Kma}{t\nu + \tau/2} V^{(l-1)} \left( \frac{\tau}{2K} - \frac{(\nu + \tau/2)x}{Kma} \right) dx \right) \\ &\ll \frac{(ma)^{j+1/2}}{\nu + \frac{\tau}{2}}. \end{aligned}$$

These calculations show that

$$\int_{\mathbb{R}} \left| \frac{d}{d\nu} \hat{g}_n(\nu) \right| d\nu \ll t^\epsilon \hat{g}_n \ll \frac{t^{\sigma+\epsilon}}{K^3 a q},$$



and by the second derivative bound for oscillatory integrals,

$$\int_{\mathbb{R}} \hat{g}_n(\nu) e(\tilde{f}(\nu)) d\nu \ll \frac{t^{1/2+\sigma+\epsilon}}{K^3 a q},$$

which is the same bound we obtained for  $x \in (0, K^{\delta'-1})$ . We therefore obtain

$$\left(\frac{aq}{t}\right)^{1/2} \int_0^1 \int_{\mathbb{R}} g(v) e(f(v)) dv \frac{dx}{x^{1/2}} = \left(\frac{aq}{t}\right)^{1/2} \int_{K^{-1+\delta'}}^1 \frac{g(v_0) e(f(v_0) + 1/8)}{x^{1/2} \sqrt{f''(v_0)}} dx + E^*,$$

where  $E^*$  is an error term such that

$$\int_{\mathbb{R}} M_t(\sigma + i\nu) W_l(\nu) t^{-i\nu} E^* d\nu \ll \frac{t^{1/2+\sigma+\epsilon}}{aqK^3}.$$

Now, plugging in the value for  $v_0$ , we get that the leading term above reduces to

$$\begin{aligned} & c_3 \frac{\nu + \frac{\tau}{2}}{K} \left(\frac{-q}{mt}\right)^{3/2} V\left(\frac{-(\nu + \frac{\tau}{2})q}{2\pi mt}\right) \left(-\frac{(\nu + \frac{\tau}{2})q}{2e\pi tm}\right)^{-i(\nu + \tau/2)} \left(\frac{tm}{(\nu + \frac{\tau}{2})q}\right)^{\sigma} \\ & \times \check{U}\left(-\frac{(\nu + \frac{\tau}{2})q}{2\pi mt}\right) \int_{K^{-1+\delta'}}^1 V\left(\frac{\tau}{2K} - \frac{(\nu + \frac{\tau}{2})x}{Kma}\right) dx, \end{aligned}$$

for some absolute constant  $c_3$ . Set

$$B(C, \tau, \nu) = t^{-5/2} + E + E^*,$$

and note that

$$(18) \quad \int_{-\frac{(tK)^{1/2}t^{\epsilon}}{C}}^{\frac{(TK)^{1/2}t^{\epsilon}}{C}} \int_{\mathbb{R}} M_t(\sigma + i\nu) W_l(\nu) t^{-i\nu} B(C, \tau, \nu) d\nu d\tau \ll \frac{t^{\sigma+\epsilon}}{t^{1/2}K^{1/2}} \left(1 + \frac{t}{C^2K^{3/2}}\right).$$

We may now derive from these computations the following:

**Lemma 8.** *We have*

$$I^{**}(q, m, \tau, \nu) = I_1(q, m, \tau, \nu) + I_2(q, m, \tau, \nu),$$

where

$$\begin{aligned} I_1(q, m, \tau, \nu) &= \frac{c_4}{(\nu + \frac{\tau}{2})^{1/2}K} \left(-\frac{(\nu + \frac{\tau}{2})q}{2e\pi tm}\right)^{3/2-i(\nu + \frac{\tau}{2})} V\left(-\frac{(\nu + \frac{\tau}{2})q}{2\pi mt}\right) \left(\frac{tm}{(\nu + \frac{\tau}{2})aq}\right)^{\sigma} \\ &\times \check{U}\left(-\frac{(\nu + \frac{\tau}{2})q}{2\pi mt}\right) \int_{K^{-1+\delta'}}^1 V\left(\frac{\tau}{2K} - \frac{(\nu + \frac{\tau}{2})x}{Kma}\right) dx, \end{aligned}$$

where  $c_4$  is an absolute constant, and

$$I_2(q, m, \tau, \nu) := I^{**}(q, m, \tau, \nu) - I_1(q, m, \tau, \nu) = B(C, \tau, \nu).$$

Consequently from Lemma 7 we arrive at:

**Lemma 9.** *We have*

$$S_l(t, C) = \sum_{J \in \mathcal{J}} \{S_{1,J}(t, C) + S_{2,J}(t, C)\} + O(t^{-1000}),$$

where

$$\begin{aligned} S_{r,J}(t, C) &= \frac{iKt^{1/2}}{4\pi^2} \sum_{\pm} \sum_{n \ll Kt^{\epsilon}} \frac{\rho_f(\mp n)}{\sqrt{n}} \\ &\sum_{C < q \leq 2C} \sum_{\substack{(m,q)=1 \\ 1 \leq |m| \ll qt^{\epsilon}}} \frac{e\left(\pm \frac{n\bar{m}}{q}\right)}{aq} I_{r,J,\pm}(q, m, n), \end{aligned}$$

and

$$I_{r,J,\pm}(q, m, n) = \int_{\mathbb{R}^2} M_t(\sigma + i\nu) W_l(\nu) t^{-i\nu} \left( \frac{2\pi\sqrt{nt}}{q} \right)^{-i\tau} \gamma_{\pm}(i\tau + 1) I_r(q, m, \tau, \nu) W_J(\tau) d\tau d\nu.$$

**4.4. Application of Cauchy and Poisson I.** We will estimate here

$$\tilde{S}_2(t, C) := \sum_{J \in \mathcal{J}} S_{2,J}(t, C).$$

Taking a dyadic subdivision and using the bound  $|\gamma_{\pm}(i\tau + 1)| \ll 1$ , we get

$$\begin{aligned} \tilde{S}_2(t, C) &\ll K t^{1/2} \int_{-\frac{(tK)^{1/2}t^\epsilon}{C}}^{\frac{(tK)^{1/2}t^\epsilon}{C}} \sum_{\pm} \sum_{\substack{1 \leq L \ll Kt^\epsilon \\ L \text{ dyadic}}} \sum_{n \in \mathbb{Z}} \frac{|\rho_f(\mp n)|}{\sqrt{n}} U\left(\frac{n}{L}\right) \\ &\quad \times \left| \sum_{C < q \leq 2C} \sum_{\substack{(m,q)=1 \\ 1 \leq |m| \ll qt^\epsilon}} \frac{e\left(\pm \frac{n\bar{m}}{q}\right)}{aq^{1-i\tau}} B(C, \tau) \right| d\tau, \end{aligned}$$

where

$$\begin{aligned} B(C, \tau) &:= \int_{\mathbb{R}} M_t(\sigma + i\nu) W_l(\nu) t^{-i\nu} B(C, \tau, \nu) d\nu \\ &\ll t^{\sigma+\epsilon} \left( \frac{1}{t^{1/2} K^{3/2}} \min \left\{ 1, \frac{10K}{|\tau|} \right\} + \frac{1}{CK^{5/2}} \right) \end{aligned}$$

By Cauchy and Rankin-Selberg, we get

$$\tilde{S}_2(t, C) \ll K t^{1/2+\epsilon} \int_{-\frac{(tK)^{1/2}t^\epsilon}{C}}^{\frac{(tK)^{1/2}t^\epsilon}{C}} \sum_{\pm} \sum_{\substack{1 \leq L \ll Kt^\epsilon \\ L \text{ dyadic}}} L^{1/2} [S_{2,\pm}(t, C, L, \tau)]^{1/2} d\tau,$$

where

$$S_{2,\pm}(t, C, L, \tau) = \sum_{n \in \mathbb{Z}} \frac{1}{n} U\left(\frac{n}{L}\right) \left| \sum_{C < q \leq 2C} \sum_{\substack{(m,q)=1 \\ 1 \leq |m| \ll qt^\epsilon}} \frac{e\left(\pm \frac{n\bar{m}}{q}\right)}{aq^{1-i\tau}} B(C, \tau) \right|^2.$$

Opening the absolute square and interchanging the order of summation, we obtain

$$S_{2,\pm}(t, C, L, \tau) = \sum_{C < q, q' \leq 2C} \sum_{\substack{(m,q)=1 \\ 1 \leq |m| \ll qt^\epsilon}} \sum_{\substack{(m',q')=1 \\ 1 \leq |m'| \ll q't^\epsilon}} \frac{|B(C, \tau)|^2}{aa'q^{1-i\tau}q'^{1+i\tau}} T,$$

where

$$T := \sum_{n \in \mathbb{Z}} \frac{1}{n} U\left(\frac{n}{L}\right) e\left(\pm \frac{n\bar{m}}{q} \mp \frac{n\bar{m}'}{q'}\right).$$

Splitting in congruence classes mod  $qq'$  and applying Poisson summation, we get

$$T = \sum_{n \in \mathbb{Z}} \delta_{\pm q' \bar{m} \mp q \bar{m}' + n \equiv 0 \pmod{qq'}} \int_{\mathbb{R}} \frac{1}{y} U(y) e\left(-\frac{Lny}{qq'}\right) dy.$$

We may now truncate the  $n$ -sum to  $n \ll C^2 t^\epsilon / L$ , for otherwise the oscillatory integral is negligibly small. We may therefore estimate

$$\begin{aligned} S_{2,\pm}(t, C, L, \tau) &\ll \sum_{C < q, q' \leq 2C} \sum_{\substack{(m,q)=1 \\ 1 \leq |m| \ll qt^\epsilon}} \sum_{\substack{(m',q')=1 \\ 1 \leq |m'| \ll q't^\epsilon}} \sum_{\substack{n \ll \frac{C^2 t^\epsilon}{L} \\ n \equiv \pm q\bar{m}' \mp q'\bar{m} \pmod{qq'}}} \frac{K|B(C, \tau)|^2}{tC^2} \\ &\ll \frac{t^\epsilon C^3 K|B(C, \tau)|^2}{tL}. \end{aligned}$$

Thus, by (18), we have

$$\begin{aligned} \tilde{S}_2(t, C) &\ll \sum_{\substack{1 \leq L \ll Kt^\epsilon \\ L \text{ dyadic}}} C^{3/2} K^{3/2} t^\epsilon \int_{-\frac{(tK)^{1/2} t^\epsilon}{C}}^{\frac{(tK)^{1/2} t^\epsilon}{C}} |B(C, \tau)| d\tau \\ &\ll t^{\sigma+\epsilon} \left( \frac{C^{3/2} K}{t^{1/2}} + \frac{t^{1/2}}{C^{1/2} K^{1/2}} \right). \end{aligned}$$

The contribution of  $S_2(t, C)$  to  $S_t^+(t)$  is therefore bounded by

$$t^\epsilon \left( \frac{t^{5/4}}{K^{3/4}} + \frac{t^{3/2}}{K^{3/2}} \right).$$

Upon taking  $K = t^{1/2}$ , we note that this is bounded by  $t^{1-1/8+\epsilon}$ .

**4.5. Application of Poisson and Cauchy II.** The analysis for  $S_{1,J}$  is more delicate as we need to exploit some cancelation coming from both the  $\nu$  and  $\tau$  integrals. The idea is to use Cauchy and Rankin-Selberg as before, but keeping the integrals over  $\tau$  and  $\nu$  inside. We may bound

$$S_{1,J}(t, C) \ll K t^{1/2} \sum_{\pm} \sum_{\substack{1 \leq L \ll Kt^\epsilon \\ L \text{ dyadic}}} L^{1/2} [S_{1,J,\pm}(t, C, L)]^{1/2},$$

where

$$\begin{aligned} S_{1,J,\pm}(t, C, L) &= \sum_{n \in \mathbb{Z}} \frac{1}{n} U\left(\frac{n}{L}\right) \left| \int_{\mathbb{R}} \int_{\mathbb{R}} M_t(\sigma + i\nu) W_t(\nu) t^{-i\nu} (2\pi\sqrt{nt})^{-i\tau} \right. \\ &\quad \left. \times \gamma_{\pm}(i\tau + 1) \sum_{C < q \leq 2C} \sum_{\substack{(m,q)=1 \\ 1 \leq |m| \ll qt^\epsilon}} \frac{e\left(\pm \frac{n\bar{m}}{q}\right)}{aq^{1-i\tau}} I_1(q, m, \tau, \nu) W_J(\tau) d\tau d\nu \right|^2. \end{aligned}$$

Opening the absolute square and interchanging the order of summation, we find that  $S_{1,J,\pm}(t, C, L)$  is given by

$$\begin{aligned} &\int_{\mathbb{R}^4} M_t(\sigma + i\nu) \overline{M_t(\sigma + i\nu')} W_t(\nu) W_t(\nu') t^{i(\frac{\tau'-\tau}{2} + \nu' - \nu)} \gamma_{\pm}(1 + i\tau) \overline{\gamma_{\pm}(1 + i\tau')} W_J(\tau) W_J(\tau') \\ &\sum_{C < q, q' \leq 2C} \sum_{\substack{(m,q)=1 \\ 1 \leq |m| \ll qt^\epsilon}} \sum_{\substack{(m',q')=1 \\ 1 \leq |m'| \ll q't^\epsilon}} \frac{I_1(q, m, \tau, \nu) \overline{I_1(q', m', \tau', \nu')}}{aa'(2\pi)^{i(\tau-\tau')} q^{1-i\tau} q'^{1+i\tau'}} \mathcal{T}' d\tau d\tau' d\nu d\nu', \end{aligned}$$

where

$$\mathcal{T}' = \sum_{n \in \mathbb{Z}} \frac{1}{n^{1+i\frac{\tau-\tau'}{2}}} U\left(\frac{n}{L}\right) e\left(\pm \frac{n\bar{m}}{q} \mp \frac{n\bar{m}'}{q'}\right).$$

Applying Poisson summation, similarly to the previous section, we obtain

$$\mathcal{T}' = \frac{L^{i\frac{\tau'-\tau}{2}}}{qq'} \sum_{n \in \mathbb{Z}} \delta \pm(n, m, m', q, q') U^\dagger\left(\frac{nL}{qq'}, -i\frac{\tau-\tau'}{2}\right),$$

where

$$\delta_{\pm}(n, m, m', q, q') = qq' \delta_{\pm q' \bar{m} \mp q \bar{m}' + n \equiv 0 \pmod{qq'}}.$$

Since  $|\tau - \tau'| \ll (tK)^{1/2} t^{\epsilon}/C$  and  $q, q' \asymp C$ , we have by Lemma 3 that if  $|n| \gg C(tK)^{1/2} t^{\epsilon}/L$ , then the contribution is negligibly small.

**Lemma 10.** *The sum  $S_{1,J,\pm}(t, C, L)$  is dominated by the sum*

$$\frac{K}{tC^2} \sum_{C < q, q' \leq 2C} \sum_{\substack{(m, q)=1 \\ 1 \leq |m| \ll qt^{\epsilon}}} \sum_{\substack{(m', q')=1 \\ 1 \leq |m'| \ll q't^{\epsilon}}} \sum_{\substack{|n| \leq C(tK)^{1/2} t^{\epsilon}/L \\ n \equiv \pm q \bar{m}' \mp q' \bar{m} \pmod{qq'}}} |\mathcal{K}_{\pm}| + O(t^{-1000}),$$

where

$$\begin{aligned} \mathcal{K}_{\pm} = & \int_{\mathbb{R}^4} M_t(\sigma + i\nu) \overline{M_t(\sigma + i\nu')} W_l(\nu) W_l(\nu') t^{i(\nu' - \nu)} \frac{(4\pi^2 tL)^{-i\frac{\tau - \tau'}{2}}}{q^{-i\tau} q'^{i\tau'}} W_J(\tau) W_J(\tau') \\ & \gamma_{\pm}(i\tau + 1) \overline{\gamma_{\pm}(i\tau' + 1)} I_1(q, m, \tau, \nu) \overline{I_1(q', m', \tau', \nu')} U^{\dagger} \left( \frac{nL}{qq'}, i\frac{\tau' - \tau}{2} \right) d\tau d\tau' d\nu d\nu'. \end{aligned}$$

We are thus only left with understanding  $\mathcal{K}_{\pm}$ . Writing out explicitly  $I_1(q, m, \tau, \nu)$ , we obtain

$$\begin{aligned} \mathcal{K}_{\pm} = & \frac{|c_4|^2}{K^2} \int_{\mathbb{R}^4} W_J(q, m, \tau, \nu) \overline{W_J(q', m', \tau', \nu')} e(f_t(\sigma + i\nu) - f_t(\sigma + i\nu')) \\ & \times t^{i(\nu' - \nu)} U^{\dagger} \left( \frac{nL}{qq'}, i\frac{\tau' - \tau}{2} \right) \left( -\frac{(\nu + \frac{\tau}{2})q}{2\pi etm} \right)^{-i(\nu + \frac{\tau}{2})} \left( -\frac{(\nu' + \frac{\tau'}{2})q'}{2\pi etm'} \right)^{i(\nu' + \frac{\tau'}{2})} \\ & \times \gamma_{\pm}(1 + i\tau) \overline{\gamma_{\pm}(1 + i\tau')} \frac{(4\pi^2 tL)^{i\frac{\tau' - \tau}{2}}}{q^{-i\tau} q'^{i\tau'}} d\tau d\tau' d\nu d\nu', \end{aligned}$$

where

$$\begin{aligned} W_J(q, m, \tau, \nu) = & g_t(\sigma + i\nu) \frac{W_l(\nu) W_J(\tau)}{(\nu + \frac{\tau}{2})^{1/2}} \left( -\frac{(\nu + \frac{\tau}{2})q}{2\pi etm} \right)^{3/2} V \left( -\frac{(\nu + \frac{\tau}{2})q}{2\pi etm} \right) \\ & \times \left( \frac{tm}{(\nu + \frac{\tau}{2})q} \right)^{\sigma} \tilde{U} \left( -\frac{(\nu + \frac{\tau}{2})q}{2\pi etm} \right) \int_{K^{\sigma'-1}}^1 V \left( \frac{\tau}{2K} - \frac{(\nu + \frac{\tau}{2})x}{Kma} \right) dx. \end{aligned}$$

We note in passing the following estimates

$$(19) \quad \frac{d}{d\tau} W_J(q, m, \tau, \nu) \ll \frac{|\nu|^{\sigma-1}}{|\tau|},$$

and

$$(20) \quad \frac{d}{d\nu} W_J(q, m, \tau, \nu) \ll |\nu|^{\sigma-2}.$$

We first analyse the case  $n = 0$ ; it will be sufficient to consider

$$\begin{aligned} & \int_{\mathbb{R}^2} W_J(q, m, \tau, \nu) \overline{W_J(q', m', \tau', \nu')} e(f_t(\sigma + i\nu) - f_t(\sigma + i\nu')) t^{i(\nu' - \nu)} \\ & \times \left( -\frac{(\nu + \frac{\tau}{2})q}{2\pi etm} \right)^{-i(\nu + \frac{\tau}{2})} \left( -\frac{(\nu' + \frac{\tau'}{2})q'}{2\pi etm'} \right)^{i(\nu' + \frac{\tau'}{2})} d\nu d\nu' \\ & = \int_{\mathbb{R}^2} W_J(q, m, \tau, \nu) \overline{W_J(q', m', \tau', \nu')} e(f(\nu, \nu')) d\nu d\nu', \end{aligned}$$

where we temporarily define

$$f(\nu, \nu') = f_t(\sigma + i\nu) - f_t(\sigma + i\nu') + \frac{\nu' - \nu}{2\pi} \log t \\ - \frac{\nu + \frac{\tau}{2}}{2\pi} \log \left( -\frac{(\nu + \frac{\tau}{2})q}{2\pi etm} \right) + \frac{\nu' + \frac{\tau'}{2}}{2\pi} \log \left( -\frac{(\nu' + \frac{\tau'}{2})q'}{2\pi etm'} \right).$$

We compute

$$\frac{df}{d\nu} = f'_t(\sigma + i\nu) - \frac{\log t}{2\pi} - \frac{1}{2\pi} \log \left( -\frac{(\nu + \frac{\tau}{2})q}{2\pi etm} \right) - \frac{1}{2\pi}, \\ \frac{df}{d\nu'} = -f'_t(\sigma + i\nu') + \frac{\log t}{2\pi} + \frac{1}{2\pi} \log \left( -\frac{(\nu' + \frac{\tau'}{2})q'}{2\pi etm'} \right) + \frac{1}{2\pi}.$$

and thus

$$\frac{d^2 f}{d\nu d\nu'} = 0,$$

while by (8), we have

$$\frac{d^2 f}{d\nu^2} = f''_t(\sigma + i\nu) - \frac{1}{2\pi(\nu + \frac{\tau}{2})} \gg |\nu|^{-1},$$

and

$$\frac{d^2 f}{d\nu'^2} = -f''_t(\sigma + i\nu') + \frac{1}{2\pi(\nu' + \frac{\tau'}{2})} \gg |\nu'|^{-1}.$$

We also note that by (20), we have

$$\text{Var}(W_J(q, m, \tau, \nu) \overline{W_J(q', m', \tau', \nu')}) \ll t^{2\sigma-2+\epsilon}.$$

We now have by the second derivative bound for oscillatory integrals in multivariables (see [10]) that

$$(21) \quad \int_{\mathbb{R}^2} W_J(q, m, \tau, \nu) \overline{W_J(q', m', \tau', \nu')} e(f(\nu, \nu')) d\nu d\nu' \ll t^{2\sigma-1+\epsilon}.$$

By integration by parts, if  $|\tau - \tau'| \gg t^\epsilon$ , then  $U^\dagger \left( 0, i\frac{\tau - \tau'}{2} \right)$  is negligibly small. The contribution from  $n = 0$  to  $\mathcal{K}_\pm$  is thus bounded by

$$K^{-2} \int \int_{\substack{|\tau - \tau'| \ll t^\epsilon \\ |\tau|, |\tau'| \asymp J}} t^{2\sigma-1+\epsilon} \ll \frac{t^{2\sigma+\epsilon}}{C t^{1/2} K^{3/2}}.$$

We now treat the case  $n \neq 0$ . We have by Lemma 5 of [8] that

$$U^\dagger \left( \frac{nL}{qq'}, -i\frac{\tau - \tau'}{2} \right) = \frac{c_5}{(\tau' - \tau)^{1/2}} U \left( \frac{(\tau' - \tau)qq'}{4\pi nL} \right) \left( \frac{(\tau' - \tau)qq'}{4\pi enL} \right)^{-i(\tau - \tau')/2} \\ + O \left( \min \left\{ \frac{1}{|\tau - \tau'|^{3/2}}, \frac{C^3}{(|n|L)^{3/2}} \right\} \right),$$

for some constant  $c_5$  (which depends on the sign of  $n$ ). In order to bound the error term, we use (21) to see that the contribution is bounded by

$$\frac{t^{2\sigma-1+\epsilon}}{K^2} \int_{[J, 2J]^2} \min \left\{ \frac{1}{|\tau - \tau'|^{3/2}}, \frac{C^3}{(|n|L)^{3/2}} \right\}.$$

We first estimate

$$\frac{t^{2\sigma-1+\epsilon}}{K^2} \int_{\substack{[J, 2J]^2 \\ |\tau - \tau'| \leq |nL|/C^2}} \frac{C^3}{(|n|L)^{3/2}} d\tau d\tau' \ll \frac{t^{2\sigma-1+\epsilon} C J}{K^2 (|n|L)^{1/2}} \ll \frac{t^{2\sigma-1/2+\epsilon}}{K^{3/2} (|n|L)^{1/2}},$$

and then

$$\begin{aligned} \frac{t^{2\sigma-1+\epsilon}}{K^2} \int_{\substack{[J,2J]^2 \\ |\tau-\tau'| > |nL|/C^2}} \frac{1}{|\tau-\tau'|^{3/2}} d\tau d\tau' &\ll \frac{Ct^{2\sigma-1+\epsilon}}{K^2(|nL|)^{1/2}} \int_{[J,2J]^2} \frac{1}{|\tau-\tau'|^{1-\epsilon}} d\tau d\tau' \\ &\ll \frac{CJt^{2\sigma-1+\epsilon}}{K^2(|nL|)^{1/2}} \ll \frac{t^{2\sigma-1/2+\epsilon}}{K^{3/2}(|nL|)^{1/2}}. \end{aligned}$$

We thus set

$$B^*(C, 0) = \frac{t^{2\sigma+\epsilon}}{K^{3/2}Ct^{1/2}},$$

and for  $n \neq 0$ ,

$$B^*(C, n) = \frac{t^{2\sigma+\epsilon}}{K^{3/2}t^{1/2}(|n|L)^{1/2}}.$$

We now consider the main term. As noted in Section 4.1, the contribution from  $\gamma_+$  is simpler, and thus we will only focus on  $\gamma_-$ . We first note that by Fourier inversion, we have

$$\left( \frac{4\pi nL}{(\tau' - \tau)qq'} \right)^{1/2} U \left( \frac{(\tau' - \tau)qq'}{4\pi nL} \right) = \int_{\mathbb{R}} U^\dagger \left( r, \frac{1}{2} \right) e \left( r \frac{(\tau' - \tau)qq'}{4\pi nL} \right) dr.$$

Pulling out the oscillation from the  $\gamma_-$  factors, we conclude that for some constant  $c_6$  (depending on the sign of  $n$ ), we have

$$\begin{aligned} \mathcal{K}_- &= \frac{c_6}{K^2} \left( \frac{qq'}{|n|L} \right)^{1/2} \int_{\mathbb{R}} U^\dagger \left( r, \frac{1}{2} \right) \int_{\mathbb{R}^4} g(\tau, \tau', \nu, \nu') e(f(\tau, \tau', \nu, \nu', r)) d\tau d\tau' d\nu d\nu' dr \\ &\quad + O(B^*(C, n)), \end{aligned}$$

where

$$\begin{aligned} f(\tau, \tau', \nu, \nu', r) &= f_t(\sigma + i\nu) - f_t(\sigma + i\nu') + \frac{\nu' - \nu}{2\pi} \log t + \frac{\tau}{2\pi} \log \left( \frac{|\tau|}{e} \right) - \frac{\tau'}{2\pi} \log \left( \frac{|\tau'|}{e} \right) \\ &\quad + \frac{\tau' - \tau}{4\pi} \log \left( \frac{(\tau' - \tau)4\pi tqq'}{en} \right) - \frac{\nu + \frac{\tau}{2}}{2\pi} \log \left( -\frac{(\nu + \frac{\tau}{2})q}{2\pi etm} \right) + \frac{\tau}{2\pi} \log q \\ &\quad - \frac{\tau'}{2\pi} \log q' + \frac{\nu' + \frac{\tau'}{2}}{2\pi} \log \left( -\frac{(\nu' + \frac{\tau'}{2})q'}{2\pi etm'} \right) + \frac{r(\tau' - \tau)qq'}{4\pi nL}, \end{aligned}$$

and

$$g(\tau, \tau', \nu, \nu') = W_J(q, m, \tau, \nu) \overline{W_J(q', m', \tau', \nu')} \Phi_-(\tau) \overline{\Phi_-(\tau')}.$$

We will use the second derivative bound for multivariable oscillatory integrals as can be found in [10] and hence compute

$$\begin{aligned} 2\pi \frac{d^2 f}{d\tau^2} &= \frac{1}{\tau} - \frac{1}{4(\nu + \frac{\tau}{2})} + \frac{1}{2(\tau' - \tau)}, \quad 2\pi \frac{d^2 f}{d\tau d\tau'} = \frac{1}{2(\tau - \tau')}, \quad 2\pi \frac{d^2 f}{d\tau d\nu} = -\frac{1}{2(\nu + \frac{\tau}{2})}, \\ 2\pi \frac{d^2 f}{d\tau'^2} &= -\frac{1}{\tau'} + \frac{1}{4(\nu' + \frac{\tau'}{2})} + \frac{1}{2(\tau' - \tau)}, \quad 2\pi \frac{d^2 f}{d\tau' d\nu'} = \frac{1}{2(\nu' + \frac{\tau'}{2})}, \\ \frac{d^2 f}{d\nu^2} &= f_t''(\sigma + i\nu) - \frac{1}{2\pi(\nu + \frac{\tau}{2})}, \quad \frac{d^2 f}{d\nu'^2} = \frac{1}{2\pi(\nu' + \frac{\tau'}{2})} - f_t''(\sigma + i\nu'), \end{aligned}$$

while

$$\frac{d^2 f}{d\tau d\nu'} = \frac{d^2 f}{d\tau' d\nu} = \frac{d^2 f}{d\nu d\nu'} = 0.$$

Computing the minors of the Hessian matrix, we see from [10, Lemma 5] that for  $D$  a box in  $R^4$ ,

$$(22) \quad \int_D e(f(\tau, \tau', \nu, \nu')) d\tau d\tau' d\nu d\nu' \ll t^\epsilon Jt,$$

where we used  $r_1 = r_2 = J^{-1/2}$  and  $r_3 = r_4 = t^{-1/2}$  as can be seen from our calculations of the second derivatives and that  $\tau, \tau' \in [J, \frac{4}{3}J]$ . Using (19) and (20), we compute the total variation, using that  $t^{1-\epsilon} \ll |\nu| \ll t^{1+\epsilon}$ :

$$\begin{aligned}
 \text{Var}(g(\tau, \tau', \nu, \nu')) &:= \int_{\mathbb{R}^4} \left| \frac{dg}{d\tau d\tau' d\nu d\nu'} \right| d\tau d\tau' d\nu d\nu' \\
 &\ll \int_{\mathbb{R}^4} \frac{|\nu|^{\sigma-2} |\nu'|^{\sigma-2}}{|\tau| |\tau'|} J t^{1+\epsilon} d\tau d\tau' d\nu d\nu' \\
 &\ll t^{2\sigma-2+\epsilon}.
 \end{aligned}
 \tag{23}$$

By integration by parts, we note that by (22) and (23), we have

$$\begin{aligned}
 &\int_{\mathbb{R}^4} g(\tau, \tau', \nu, \nu') e(f(\tau, \tau', \nu, \nu', r)) d\tau d\tau' d\nu d\nu' \\
 &\ll \int_{\mathbb{R}^4} \left| \frac{dg}{d\tau d\tau' d\nu d\nu'} \right| J t^{1+\epsilon} d\tau d\tau' d\nu d\nu' \\
 &\ll J t^{2\sigma-1+\epsilon}.
 \end{aligned}$$

Then, integrating trivially over  $r$  and using the rapid decay of Fourier transforms, we arrive at the following result:

**Lemma 11.** *We have*

$$\mathcal{K}_- \ll B^*(C, n).$$

We now write

$$S_{1,J,-}(t, C, L) = S_{1,J,-}^{\flat}(t, C, L) + S_{1,J,-}^{\sharp}(t, C, L),$$

where  $S_{1,J,-}^{\flat}(t, C, L)$  corresponds to  $n = 0$  contribution, while  $S_{1,J,-}^{\sharp}(t, C, L)$  corresponds to the  $n \neq 0$  frequencies. We first estimate

$$\begin{aligned}
 S_{1,J,-}^{\flat}(t, C, L) &\ll \frac{K}{tC^2} \sum_{C < q, q' \leq 2C} \sum_{\substack{(m,q)=1 \\ 1 \leq |m| \ll qt^\epsilon}} \sum_{\substack{(m',q')=1 \\ 1 \leq |m'| \ll q't^\epsilon}} \delta_{-q\overline{m'} + q'\overline{m} \equiv 0 \pmod{qq'}} \frac{t^{2\sigma+\epsilon}}{K^{3/2} C t^{1/2}} \\
 &\ll \frac{t^{2\sigma+\epsilon}}{t^{3/2} K^{1/2}}.
 \end{aligned}$$

Taking a dyadic subdivision, we estimate

$$\begin{aligned}
S_{1,J,-}^\#(t, C, L) &\ll \frac{K}{tC^2} \sum_{C < q, q' \leq 2C} \sum_{\substack{(m,q)=1 \\ 1 \leq |m| \ll qt^\epsilon}} \sum_{\substack{(m',q')=1 \\ 1 \leq |m'| \ll q't^\epsilon}} \sum_{\substack{1 \leq |n| \ll \frac{C(tK)^{\frac{1}{2}} t^\epsilon}{L} \\ n \equiv -qm' + q'\overline{m} \pmod{qq'}}} B^*(C, n) \\
&\ll \frac{K}{tC^2} \sum_{C < q, q' \leq 2C} \sum_{\substack{(m,q)=1 \\ 1 \leq |m| \ll qt^\epsilon}} \sum_{\substack{(m',q')=1 \\ 1 \leq |m'| \ll q't^\epsilon}} \sum_{\substack{1 \leq |n| \ll \frac{C(tK)^{\frac{1}{2}} t^\epsilon}{L} \\ n \equiv q'\overline{m} - qm' \pmod{qq'}}} \frac{t^{2\sigma+\epsilon}}{K^{3/2} t^{1/2} (|n|L)^{\frac{1}{2}}} \\
&\ll \frac{t^{2\sigma+\epsilon}}{t^{3/2} K^{1/2} C^2 L^{1/2}} \sum_{\substack{H \leq \frac{C(tK)^{1/2} t^\epsilon}{L} \\ H \text{ Dyadic}}} \sum_{C < q, q' \leq 2C} \sum_{\substack{(m,q)=1 \\ 1 \leq |m| \ll qt^\epsilon}} \\
&\times \sum_{\substack{(m',q')=1 \\ 1 \leq |m'| \ll q't^\epsilon}} \sum_{\substack{H < |n| \leq 2H \\ n \equiv -qm' + q'\overline{m} \pmod{qq'}}} H^{-1/2} \\
&\ll \frac{t^{2\sigma+\epsilon}}{t^{3/2} K^{1/2} C^2 L^{1/2}} \sum_{\substack{H \leq \frac{C(tK)^{1/2} t^\epsilon}{L} \\ H \text{ Dyadic}}} H^{-1/2} \sum_{C < q, q' \leq 2C} \sum_{H < |n| \leq 2H} \\
&\times \sum_{\substack{1 \leq |m| \ll qt^\epsilon \\ (m,q)=1}} \sum_{\substack{1 \leq |m'| \ll q't^\epsilon \\ (m',q')=1}} \delta_{-qm' + q'\overline{m} \equiv n \pmod{qq'}}.
\end{aligned}$$

We let  $d = (q, q')$  and notice that looking at the congruence condition above modulo  $q$  implies that  $q'\overline{m} \equiv n \pmod{q}$ , which in turn implies that  $d$  divides  $n$ . We let  $q_0 := q/d, q'_0 = q'/d$  and  $n_0 := n/d$ , so that

$$n_0 \equiv q'_0 \overline{m} \pmod{q_0}, \text{ and } n_0 \equiv q_0 \overline{m'} \pmod{q'_0}.$$

We may thus bound

$$\begin{aligned}
S_{1,J,-}^\#(t, C, L) &\ll \frac{t^{2\sigma+\epsilon}}{t^{3/2} K^{1/2} C^2 L^{1/2}} \sum_{\substack{H \leq \frac{C(tK)^{1/2} t^\epsilon}{L} \\ H \text{ Dyadic}}} H^{-1/2} \sum_{C < q, q' \leq 2C} \sum_{\frac{H}{d} < n_0 \leq \frac{2H}{d}} \\
&\times \sum_{\substack{1 \leq |m| \ll qt^\epsilon \\ (m,q)=1}} \sum_{\substack{1 \leq |m'| \ll q't^\epsilon \\ (m',q')=1}} \delta_{q'_0 \overline{m} \equiv n_0 \pmod{q_0}} \delta_{q_0 \overline{m'} \equiv n_0 \pmod{q'_0}} \\
&\ll \frac{t^{2\sigma+\epsilon}}{t^{3/2} K^{1/2} C^2 L^{1/2}} \sum_{\substack{H \leq \frac{C(tK)^{1/2} t^\epsilon}{L} \\ H \text{ Dyadic}}} H^{-1/2} \sum_{C < q, q' \leq 2C} \sum_{\frac{H}{d} < n_0 \leq \frac{2H}{d}} t^\epsilon d^2 \\
&\ll \frac{t^{2\sigma+\epsilon} (tK)^{1/4}}{t^{3/2} K^{1/2} C^{3/2} L} \sum_{C < q, q' \leq 2C} d \\
&\ll \frac{t^{2\sigma+\epsilon} (tK)^{1/4}}{t^{3/2} K^{1/2} C^{3/2} L} \sum_{d \leq 2C} \sum_{\frac{C}{d} \leq q_0, q'_0 \leq \frac{2C}{d}} d \\
&\ll \frac{t^{2\sigma+\epsilon}}{tK^{1/2} L}.
\end{aligned}$$

We conclude that

$$S_{1,J,-}(t, C, L) \ll t^{2\sigma+\epsilon} \left( \frac{1}{t^{3/2} K^{1/2}} + \frac{1}{tK^{1/2} L} \right).$$



The same bound holds for  $S_{1,J,+}(t, C, L)$ , via the same analysis, so that

$$\begin{aligned} S_{1,J}(t, C) &\ll K t^{\sigma+1/2+\epsilon} \sum_{\substack{1 \leq L \ll K t^\epsilon \\ L \text{ Dyadic}}} \left( \frac{L^{1/2}}{t^{3/4} K^{1/4}} + \frac{1}{t^{1/2} K^{1/4}} \right) \\ &\ll t^{\sigma+\epsilon} \left( \frac{K^{5/4}}{t^{1/4}} + K^{3/4} \right). \end{aligned}$$

The same bound holds for all values of  $J$ . Since there are  $O(\log t)$  many terms, we can sum over them without worsening the bound, and so the same bound holds for  $\hat{S}_1(t, C) := \sum_J S_{1,J}(t, C)$ . Thus the total contribution of  $\hat{S}_1(t, C)$  to  $S_l^+(t)$  is bounded by

$$\frac{t^{1+\epsilon}}{K} \left( \frac{K^{5/4}}{t^{1/4}} + K^{3/4} \right) \ll t^\epsilon \left( t^{3/4} K^{1/4} + \frac{t}{K^{1/4}} \right).$$

Choosing  $K = t^{1/2}$ , we obtain

$$S_l^+(t) \ll t^{1-1/8+\epsilon}.$$

## 5. EXAMPLES

In this section, we study some examples of analytic trace functions to motivate the analogy with Frobenius trace functions studied in [4]. The analog of Kloosterman sums is given in the following example.

**Proposition 2.** *Let*

$$F_{it}(x) := t^{1/2} \Gamma\left(\frac{1}{2} + it\right) J_{it}(x)$$

*be the normalized  $J$ -Bessel function of order  $t$ . Then,  $F_{it}$  is an analytic trace function.*

*Proof.* By [2, p. 331], the Mellin inversion theorem holds for  $F_{it}$  and the Mellin transform is given by

$$M_{F,t}(s) := \int_0^\infty F_{it}(x) x^{s-1} dx = t^{1/2} \Gamma\left(\frac{1}{2} + it\right) 2^{s-1} \frac{\Gamma\left(\frac{s+it}{2}\right)}{\Gamma\left(1 + \frac{it-s}{2}\right)},$$

for any  $0 < \sigma < 1$ , where  $s = \sigma + i\nu$ . We will assume for simplicity that  $t \geq 1$ , the same argument holding also for negative  $t$ . In order to understand  $M_{F,t}(\sigma + i\nu)$ , we differentiate between three cases, using Stirling's formula for some of the Gamma factors. We first note that

$$(24) \quad \left| \Gamma\left(\frac{1}{2} + it\right) \right| = \sqrt{2\pi} \exp\left(-\frac{\pi t}{2}\right) (1 + O(|t|^{-1})).$$

First assume we are in the range where  $|t \pm \nu| \geq 1$ , then we may apply Stirling's formula to all the Gamma factors, and find that

$$M_{F,t}(s) = t^{1/2} g_{F,t}(s) e(f_{F,t}(s)),$$

where, up to a constant,

$$g_{F,t}(s) = \exp\left(\frac{\pi}{4}(|t - \nu| - |\nu + t| - 2t)\right) |(\nu + t)(t - \nu)|^{\frac{\sigma-1}{2}} (1 + O(\max\{t^{-1}, |t \pm \nu|^{-1}\})),$$

and

$$2\pi f_{F,t}(s) = \frac{\nu + t}{2} \log \left| \frac{\nu + t}{2e} \right| + \frac{\nu - t}{2} \log \left| \frac{t - \nu}{2e} \right| + \nu \log 2.$$

We note that if  $\nu \geq -\frac{t}{2}$ , then  $g_{F,t}(s)$  is negligible. We therefore only focus on the case where  $\nu < -\frac{t}{2}$  and verify condition (5) for  $f_{F,t}$ . We thus compute

$$2\pi \frac{d}{d\nu} f_{F,t}(s) = \frac{1}{2} \log \left| \frac{t^2 - \nu^2}{4e^2} \right| + 1 + \log 2.$$

Since we only consider  $\nu \gg t$  by exponential decay of  $g_{F,t}$  otherwise, we find that

$$\log \left| \frac{(t^2 - \nu^2)^{1/2}}{x} \right| \ll 1,$$

may only occur if  $\nu \asymp t$ , for  $x \in [t, 2t]$ .

On the other hand, if we are in the range  $|t - \nu| < 1$ , then we may not apply Stirling's formula for the Gamma factor in the denominator. However, we will have that  $|t + \nu| \gg t$ , and thus by (24) and the exponential decay of Gamma factors, we get that the contribution is negligible. Finally, if we are in the range  $|t + \nu| < 1$ , then the phase of  $M_{F,t}(s)$  will be of the form

$$2\pi \tilde{f}_{F,t}(s) := \frac{\nu - t}{2} \log \left| \frac{t - \nu}{2e} \right| + \nu \log 2,$$

and so

$$2\pi \frac{d}{d\nu} \tilde{f}_{F,t}(s) - \log(x) \gg 1$$

in this region, and is thus negligible by integration by parts. Moreover, looking at  $f_{F,t}$ , there can be no stationary point in any region such that  $\nu = -t + o(t)$ .

We thus assume from now on that we are in the region where  $|t \pm \nu| \gg t$ , and  $t \ll \nu \leq -t$ , and will show that conditions (4), (6), (7) and (8) hold for  $g_{F,t}(s)$  and  $f_{F,t}(s)$ . Indeed, in this region,

$$t^{1/2} g_{F,t}(s) = t^{1/2} |(\nu + t)(\nu - t)|^{\frac{\sigma-1}{2}} (1 + O(t^{-1})) \ll t^{\sigma-1/2},$$

and thus

$$t^{1/2} \frac{d^j}{d\nu^j} g_{F,t}(s) \ll t^{\sigma-1/2-j},$$

for all  $j \geq 0$ , proving (4). We now compute

$$2\pi \frac{d^2}{d\nu^2} f_{F,t}(s) = \frac{\nu}{(\nu^2 - t^2)} \gg \nu^{-1},$$

and thus

$$2\pi \frac{d^j}{d\nu^j} f_{F,t}(s) \ll_{j,\epsilon} \nu^{1+\epsilon-j},$$

for all  $j \geq 0$ , proving (6) and (7). Finally we look at

$$2\pi \frac{d^2}{d\nu^2} f_{F,t}(s) - \frac{1}{\nu} = \frac{t^2}{\nu(\nu^2 - t^2)} \gg \nu^{-1},$$

proving (8), concluding the proof that  $F_{it}$  is an analytic trace function.  $\square$

Another interesting example is that of Bessel functions of high rank. These can be thought of as analogs to hyper-Kloosterman sums. We study here higher rank Bessel functions appearing in the Voronoi summation formulas in higher rank (as in [9]).

**Proposition 3.** *For any  $n \geq 3$ , let*

$$J_{n,t} := \frac{t^{\frac{n-1}{2}}}{2\pi i n} \int_{(\frac{1}{4})} \Gamma\left(\frac{s - int}{n}\right) \Gamma\left(\frac{s}{n} + \frac{it}{n-1}\right)^{n-1} e\left(\frac{s}{4}\right) x^{-s} ds.$$

*Then  $J_{n,t}$  is an analytic trace function.*

*Proof.* Let

$$M_{J_{n,t}}(s) := \frac{t^{\frac{n-1}{2}}}{n} \Gamma\left(\frac{s-int}{n}\right) \Gamma\left(\frac{s}{n} + \frac{it}{n-1}\right)^{n-1} e\left(\frac{s}{4}\right),$$

with  $s = \frac{1}{4} + i\nu$ . We assume again for simplicity that  $t > 1$  and want to show that  $M_{J_{n,t}}$  satisfies all the conditions in Definition 2. As in the case of the Bessel function, we wish to use Stirling's formula to understand the phase and amplitude of  $M_{J_{n,t}}$ . Again we distinguish three different cases. First assume we are in the range  $|\nu - nt| \geq n$  and  $|(n-1)\nu + nt| \geq n(n-1)$ . We may then apply Stirling's formula to both Gamma factors to obtain

$$M_{J_{n,t}}(s) = e\left(\frac{1}{8}\right) \frac{t^{\frac{n-1}{2}}}{n} g_{J_{n,t}}(s) e(f_{J_{n,t}}(s)),$$

where  $g_{J_{n,t}}(s)$  is given by

$$\begin{aligned} & \exp\left(-\frac{\pi(|\nu - nt| + |(n-1)\nu + nt| + n\nu)}{2n}\right) \left|\frac{\nu - nt}{n}\right|^{\frac{1}{4n}-\frac{1}{2}} \left|\frac{\nu}{n} + \frac{t}{n-1}\right|^{(n-1)(\frac{1}{4n}-\frac{1}{2})} \\ & \times \left(1 + O\left((1 + |\nu - nt|)^{-1} + \left(1 + \left|\frac{\nu}{n} + \frac{t}{n-1}\right|\right)^{-1}\right)\right), \end{aligned}$$

and

$$2\pi f_{J_{n,t}}(s) = \frac{(n-1)\nu + nt}{n} \log\left|\frac{\nu}{en} + \frac{t}{e(n-1)}\right| + \frac{\nu - nt}{n} \log\left|\frac{\nu - nt}{ne}\right|.$$

We note that if  $\nu \geq -\frac{n}{2(n-1)}t$ , then  $g_{J_{n,t}}$  is negligible. We therefore only focus on the case where  $\nu < -\frac{n}{2(n-1)}t$  and verify condition (5) for  $f_{J_{n,t}}$ . We thus compute

$$2\pi \frac{d}{d\nu} f_{J_{n,t}}(s) = \frac{n-1}{n} \log\left|\frac{\nu}{en} + \frac{t}{e(n-1)}\right| + \frac{1}{n} \log\left|\frac{\nu - nt}{ne}\right| + 1.$$

Since we only consider  $\nu \gg t$  by exponential decay of  $g_{J_{n,t}}$  otherwise, we find that

$$\log\left|\left(\frac{(n-1)\nu + nt}{n-1}\right)^{\frac{n-1}{n}} \frac{(\nu - nt)^{\frac{1}{n}}}{xn}\right| \ll 1,$$

may only occur if  $\nu \asymp x$ , for  $x \asymp t$ . Moreover, as in the Bessel function case, we see from this that in the two cases where we might not use Stirling's formula for one of the Gamma factors, either  $g_{J_{n,t}}$  will be negligible, or the phase cannot vanish and the contribution is also negligible.

We thus assume from now on that we are in the region where  $|(n-1)\nu + nt|, |\nu - nt| \gg t$  and  $t \ll \nu \leq -\frac{n}{(n-1)}t$ , and will show that conditions (4), (6), (7) and (8) hold for  $g_{J_{n,t}}(s)$  and  $f_{J_{n,t}}(s)$ . Indeed, in this region,

$$t^{\frac{n-1}{2}} g_{J_{n,t}}(s) = t^{\frac{n-1}{2}} \left|\frac{\nu - nt}{n}\right|^{\frac{1}{4n}-\frac{1}{2}} \left|\frac{\nu}{n} + \frac{t}{n-1}\right|^{(n-1)(\frac{1}{4n}-\frac{1}{2})} (1 + O(t^{-1})) \ll t^{\frac{1}{4}-\frac{1}{2}},$$

and thus

$$t^{\frac{n-1}{2}} \frac{d^j}{d\nu^j} g_{J_{n,t}}(s) \ll t^{\frac{1}{4}-\frac{1}{2}-j},$$

for all  $j \geq 0$ , proving (4). We now compute

$$2\pi \frac{d^2}{d\nu^2} f_{J_{n,t}}(s) = \frac{(n-1)\nu + nt(2-n)}{(\nu - nt)((n-1)\nu + nt)} \gg \nu^{-1},$$

since  $\nu < 0$ , and thus

$$2\pi \frac{d^j}{d\nu^j} f_{J_{n,t}}(s) \ll_{j,\epsilon} \nu^{1+\epsilon-j},$$

for all  $j \geq 0$ , proving (6) and (7). Finally, we look at

$$2\pi \frac{d^2}{d\nu^2} f_{J_{n,t}}(s) - \frac{1}{\nu} = \frac{nt^2}{\nu(\nu - nt)((n-1)\nu + nt)} \gg \nu^{-1},$$

proving (8), concluding the proof that  $J_{n,t}$  is an analytic trace function.  $\square$

We end this section with an example motivating condition (8). Namely, we study  $e(x)$  in the range  $x \in [t, 2t]$  and show that it satisfies all the conditions to be an analytic trace function, besides (8). By Mellin inversion, we thus have

$$\begin{aligned} V\left(\frac{x}{t}\right) e(x) &= \frac{1}{2\pi} \int_{\mathbb{R}} t^{i\nu} V^\dagger(-t, i\nu) x^{-i\nu} d\nu \\ &:= \frac{1}{2\pi} \int_{\mathbb{R}} M_{e,t}(i\nu) x^{-i\nu} d\nu, \end{aligned}$$

where

$$M_{e,t}(i\nu) = t^{i\nu} V^\dagger(-t, i\nu).$$

We first note that by Lemma 3, we may assume that  $\nu \asymp t$ , for otherwise  $V^\dagger(-t, i\nu)$  is negligible. We now use Lemma 5 in [8] to write in this region

$$M_{e,t}(i\nu) = g_{e,t}(i\nu) e(f_{e,t}(i\nu)),$$

where, up to a constant,

$$g_{e,t}(i\nu) = \nu^{-1/2} V\left(-\frac{\nu}{2\pi t}\right) (1 + O(\nu^{-3/2})),$$

and

$$f_{e,t}(i\nu) = \frac{\nu}{2\pi} \log\left(-\frac{\nu}{2\pi e}\right).$$

One now verifies that

$$g_{e,t}^{(j)}(i\nu) \ll_j \nu^{-1/2-j},$$

for all  $j \geq 0$ . We compute

$$f'_{e,t}(i\nu) = \frac{1}{2\pi} \log\left(-\frac{\nu}{2\pi e}\right) + \frac{1}{2\pi},$$

and

$$f_{e,t}^{(j)}(i\nu) = \frac{(-1)^j}{2\pi \nu^{j-1}},$$

for  $j \geq 2$ . We thus have that  $f_{e,t}$  satisfies (6), (7), and the only condition not satisfied is (8). Given that our results should generalise to holomorphic forms as well as Eisenstein series, this example illustrates the necessity of condition (8), since the divisor function,  $d(n)$ , correlates with additive characters [12, Theorem 7.15].

## 6. HOROCYCLE TWISTS

In this section, we prove Theorem 2. We thus let  $K_t : \mathbb{R}_{>0} \rightarrow \mathbb{C}$  be an analytic trace function, and  $f$  be a Maass form as in the previous sections. Let  $[\alpha, \beta] \subset [1, 2]$  and  $V$  be a smooth compactly supported function in  $[\frac{1}{2}, \frac{5}{2}]$ , such that  $x^j V^j(x) \ll_j 1$ . We study

$$\int_{\alpha}^{\beta} f(x + iy) K_{1/y}\left(\frac{x}{y}\right) V(x) dx = \sum_{n \neq 0} \frac{\rho_f(n)}{|n|^{1/2}} W_{it_f}(4\pi|n|y) \int_{\alpha}^{\beta} K_{1/y}\left(\frac{x}{y}\right) e(nx) V(x) dx.$$

The proof of the theorem will then follow from the following proposition.

**Proposition 4.** *Let  $K_t$  be an analytic trace function. Then there exists an analytic trace function,  $\tilde{K}_t(x)$ , such that the Fourier transform,*

$$\hat{K}_t(x) := t^{1/2} \int_1^2 K_t(tu)V(u)e(-xu)du,$$

*satisfies*

$$\hat{K}_t(x) = \tilde{K}_t(x) + O(t^{-1/2}).$$

*Proof.* We have

$$\begin{aligned} \int_1^2 K_t(tu)V(u)e(-xu)du &= \frac{1}{2\pi i} \int_{(\sigma)} M_t(s) \int_1^2 (tu)^{-s} V(u)e(-xu)du ds \\ &= \frac{1}{2\pi i} \int_{(\sigma)} M_t(s) t^{-s} V^\dagger(x, 1-s) ds. \end{aligned}$$

We note that by the properties of  $M_t(s)$ , discussed in Section 3, it is sufficient to consider  $\nu \asymp t$ , such that for some  $x \in [t, 2t]$ ,

$$(25) \quad f'_t(\sigma + i\nu) - \frac{\log x}{2\pi} = o(1),$$

for otherwise by repeated integration by parts, the integral is negligible. By Lemma 5 of [8], we may write

$$V^\dagger(x, 1-\sigma-i\nu) = \frac{\sqrt{2\pi}e(1/8)}{\sqrt{\nu}} V\left(-\frac{\nu}{2\pi x}\right) \left(-\frac{\nu}{2\pi x}\right)^{1-\sigma} \left(-\frac{\nu}{2\pi ex}\right)^{-i\nu} + O(|\nu|^{-3/2}).$$

We thus have that the main term of  $\hat{K}_t(x)$  is

$$\frac{e(1/8)t^{1/2-\sigma}}{\sqrt{2\pi}i} \int_{(\sigma)} M_t(\sigma + i\nu) W(\nu) \frac{t^{-i\nu}}{\sqrt{\nu}} V\left(-\frac{\nu}{2\pi x}\right) \left(-\frac{\nu}{2\pi x}\right)^{1-\sigma} \left(-\frac{\nu}{2\pi ex}\right)^{-i\nu} d\nu,$$

where  $W$  is a smooth compactly supported function such that  $W^{(j)}(\nu) \ll_j \nu^{-j}$ , and supported only whenever (25) holds. We may thus rewrite the main term as

$$\frac{1}{2\pi i} \int_{(1-\sigma)} \tilde{M}_{t,x}(1-\sigma+i\nu) x^{\sigma-1-i\nu} d\nu,$$

where up to a constant,

$$\tilde{M}_{t,x}(1-\sigma+i\nu) = t^{1/2-\sigma+i\nu} M_t(\sigma-i\nu) W(-\nu) V\left(\frac{\nu}{2\pi x}\right) \nu^{1/2-\sigma+i\nu} (2\pi e)^{-i\nu}.$$

We write

$$\tilde{M}_{t,x}(1-\sigma+i\nu) = \tilde{g}_{t,x}(1-\sigma+i\nu) e(\tilde{f}_t(1-\sigma+i\nu)),$$

where

$$\tilde{g}_{t,x}(1-\sigma+i\nu) = t^{1/2-\sigma} W(-\nu) g_t(\sigma-i\nu) V\left(\frac{\nu}{2\pi x}\right) \nu^{1/2-\sigma},$$

and

$$\tilde{f}_t(1-\sigma+i\nu) = \frac{\nu}{2\pi} \log(t\nu) + f_t(\sigma-i\nu).$$

We compute

$$\frac{d}{d\nu} \tilde{f}_t(1-\sigma+i\nu) - \frac{1}{2\pi} \log x = \frac{1}{2\pi} \log\left(\frac{t\nu}{2\pi x}\right) - f'_t(\sigma-i\nu),$$

and note that if  $\frac{\nu}{2\pi x} \notin [\frac{1}{2}, \frac{5}{2}]$ , then by (25), we have that (5) holds, so that by repeated integration by parts the integral in that region is negligible. We may therefore write

$$\int_{(1-\sigma)} \tilde{M}_{t,x}(1-\sigma+i\nu) x^{\sigma-1-i\nu} d\nu = \int_{(1-\sigma)} \tilde{M}_t(1-\sigma+i\nu) x^{\sigma-1-i\nu} d\nu + O(t^{-100}),$$

where  $\tilde{M}_t(1 - \sigma + i\nu) = \tilde{g}_t(1 - \sigma + i\nu)e(\tilde{f}(1 - \sigma + i\nu))$ , and

$$\tilde{g}_t(1 - \sigma + i\nu) = t^{1/2-\sigma}W(-\nu)g_t(1 - \sigma + i\nu)\nu^{1/2-\sigma}.$$

In the range  $\nu \asymp t$ , we have

$$\tilde{g}_t^{(j)}(1 - \sigma + i\nu) \ll t^{1/2-\sigma-j},$$

and therefore  $\tilde{g}_t$  satisfies condition (4). We moreover have

$$\frac{d^2}{d\nu^2}\tilde{f}_t(1 - \sigma + i\nu) = \frac{1}{2\pi\nu} + f_t''(\sigma - i\nu) \gg \nu^{-1},$$

by (8) and thus (6) is satisfied for  $\tilde{f}_t$ . Moreover, by direct computation, we see that since (7) holds for  $f_t$ , it also holds for  $\tilde{f}_t$ . By (6), we have

$$\tilde{f}_t''(1 - \sigma + i\nu) - \frac{1}{2\pi\nu} = f_t''(\sigma - i\nu) \gg \nu^{-1},$$

so that (8) holds for  $\tilde{f}_t$ .  $\square$

We deduce Theorem 2 from Proposition 4. We first note that the exponential decay of  $W_{it_f}$  restricts  $n$  to the range  $|n| \ll y^{-1}$ . Keeping in mind that the Fourier transform is negligible unless  $n \asymp y^{-1}$ , we only need to show that

$$\frac{1}{\beta - \alpha} \sum_{n \asymp y^{-1}} \frac{\rho_f(n)}{y^{1/2}|n|^{1/2}} y^{1/2} \int_{\alpha}^{\beta} K_{1/y}\left(\frac{x}{y}\right) e(nx)V(x)dx \rightarrow 0,$$

as  $y \rightarrow 0$ . However, by Fourier inversion, we have

$$\begin{aligned} y^{-1/2} \int_{\alpha}^{\beta} K_{1/y}\left(\frac{x}{y}\right) e(nx)V(x)dx &= \int_{\alpha}^{\beta} \int_{\mathbb{R}} \hat{K}_{1/y}(z)e(zx)e(nx)dzdx \\ &= \int_{\mathbb{R}} \hat{K}_{1/y}(z+n) \int_{\alpha}^{\beta} e(zx)dx dz \\ &= \frac{1}{2\pi i} \int_{\mathbb{R}} \hat{K}_{1/y}(z+n) \frac{e(\beta z) - e(\alpha z)}{z} dz. \end{aligned}$$

Now by Proposition 4 and the properties of analytic trace functions, we must have  $z + n \asymp y^{-1}$ , for otherwise  $\hat{K}_{1/y}(z+n)$  is negligible. We may thus apply Theorem 1 to conclude that

$$\frac{1}{\beta - \alpha} \sum_{n \asymp y^{-1}} \frac{\rho_f(n)}{y^{1/2}|n|^{1/2}} y^{1/2} \int_{\alpha}^{\beta} K_{1/y}\left(\frac{x}{y}\right) e(nx)V(x)dx \ll \frac{y^{1/8-\epsilon}}{\beta - \alpha},$$

proving Theorem 2.

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